A NEW BEAM ELEMENT WITH TRANSVERSAL AND WARPING EIGENMODES

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ABSTRACT:

In this work, we present a new formulation of a 3D beam element, with a new method to describe the transversal deformation of the beam cross section and its warping. With this new method we use an enriched kinematics, allowing us to overcome the classical assumptions in beam theory, which states that the plane section remains plane after deformation and the cross section is infinitely rigid in its own plane. The transversal deformation modes are determined by decomposing the cross section into 1D elements for thin walled profiles and triangular elements for arbitrary sections, and assembling its rigidity matrix from which we extracts the Eigen-pairs. For each transversal deformation mode, we determine the corresponding warping modes by using an iterative equilibrium scheme. The additional degree of freedom in the enriched kinematics will give rise to new equilibrium equations, these have the same form as for a gyroscopic system in an unstable state, these equations will be solved exactly, leading to the formulation of a mesh free element. The results obtained from this new beam finite element are compared with the ones obtained with a shell model of the beam.

Keywords: Distortion, warping, higher order beam element, mesh-free element, gyroscopic system.

1. INTRODUCTION:

The classical beam theories are all based on some hypothesis that are sufficient in most cases for structure analysis, but fail in more complex cases to give accurate results and can lead to non-negligible errors. For Timoshenko beam theory, widely used by structural engineers, two assumptions are made, the cross section remains plane after deformation and every section is infinitely rigid in its own plane, this means that the effects of warping shear lag and transversal deformation are neglected, these phenomenon are important in bridge study, especially when dealing with bridge with small width/span ratio, and with thin walled cross section.

The problem of introducing the warping effect into beam theory has been widely treated. The most classical approach is to introduce extra generalized coordinates, associated with the warping functions calculated from the Saint-Venant solution, which is exact for the uniform warping of a beam, but gives poor results in the inverse case, especially near the perturbation where the warping is restrained. Bauchau[1], proposes an approach that consists in improving the Saint-Venant solution, that considers only the warping modes for a uniform warping, by adding new eigenwarping modes, derived from the principle of minimum potential energy. Sapountzakis and Mokos[2,3] calculate a secondary shear stress, due to a non-uniform torsion warping, this can be considered as the derivation of the second torsion warping mode in the work of Ferradi et al[4], where a more general formulation is given, based on a kinematics with multiple warping eigenmodes, obtained by considering an iterative equilibrium scheme, where at each iteration, equilibrating the residual warping normal stress will lead to the determination of the next mode, this method has given very accurate results, even in the vicinity of a fixed end where the condition of no warping is imposed.

The aim of this paper is to propose a new formulation, which not only takes into account the warping of the cross section, but also its transversal deformation, an element of this type falls in the category of GBT(generalized beam theory), which is essentially used to study elastic buckling of thin walled beam and cold formed steel members [5], this is done by enriching the beam's kinematics with transversal deformation modes, and then determining the contribution of every modes to the vibration of the beam. Free software GBTUL [6] is available to perform this analysis, developed by a research group at the Technical University of Lisbon. In the formulation developed by Ferradi et al[7], a series of warping functions are determined, associated to the three rigid body motions of horizontal and vertical displacements and torsion, which can be considered as the three first transversal deformation modes. The idea is to go beyond these three first modes, and determine a series of new transversal deformation modes, calculated for an arbitrary cross section, by modeling this section with triangular or/and 1D element, assembling its rigidity matrix and extracting the eigenvalues and the corresponding eigenvectors, for a desired number of modes. Then, for each determined mode, we will derive a series of warping functions, noting that we will need at least one to represent exactly the case of uniform warping in the beam. With all these additional transversal and warping modes, we will obtain an enhanced kinematics, capable of describing accurately, arbitrary displacement and stress distribution in the beam. Using the principal of virtual work we will derive the new equilibrium equations, which appear to have the same form as the dynamical equations of a gyroscopic system in an unstable state. Unlike classical finite element formulation, where interpolation functions are used for the generalized coordinates, we will perform for this formulation, as in [4], an exact solution for the arising differential equations system, leading to the formulation of a completely mesh free element.

The results obtained from the beam element will be compared to those obtained from a shell (MITC-4) and a brick (SOLID186 in Ansys™) model of the beam. Different examples are presented to illustrate the efficiency and the accuracy of this formulation.

2. DETERMINATION OF TRANSVERSAL DEFORMATION MODES:

For an arbitrary beam cross section, composed of multiple contours and thin walled profiles, to calculate the transversal deformation mode, we use a mesh with triangular elements for the 2D domain delimited by some contours and beam element for the thin walled profiles. As for a classical structure with beam and shell elements, we assemble the rigidity matrix $\boldsymbol{K_s}$ for the section, by associating to each triangular and beam element a rigidity matrix (see appendix A1) calculated for a given thickness. We calculate the eigenvalues and their associated eigenvectors of the assembled rigidity matrix of the section, by solving the standard eigenvalue problem (SEP):

$$\boldsymbol{K}_s\boldsymbol{v} = \lambda\boldsymbol{v} \qquad (1)\ \text{a}$$

We note that for all that will follow, if it’s written in bold, a lowercase letter means a vector and an uppercase letter means a matrix.

The strain energy associated to a transversal mode represented by its Eigen-pair $(\lambda, \boldsymbol{v})$ will be given by:

$$U = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{K}_s\boldsymbol{v} = \frac{1}{2}\lambda\boldsymbol{v}^T\boldsymbol{v} = \frac{1}{2}\lambda \qquad (1)\ \text{b}$$

Thus, the modes with the lowest eigenvalues mobilize less energy and then have more chances to occur. From the resolution of the SEP, we obtain a set of vectors that we note $\boldsymbol{\psi}^i = \left(\psi_y^i\ ,\ \psi_z^i\right)$, where ψ_y^i and ψ_z^i are the vertical and horizontal displacement, respectively, for the i^th transversal deformation mode. We note that the three first modes with a zero eigenvalue, corresponds to the classical modes of a rigid body motion:

$$\boldsymbol{\psi}^1 = (1\ ,0)\quad ,\quad \boldsymbol{\psi}^2 = (0\ ,1)\quad ,\quad \boldsymbol{\psi}^3 = (-(z - z_0)\ , y - y_0) \tag{2}$$

Where (y_0, z_0) are the coordinates of the torsion center of the section.

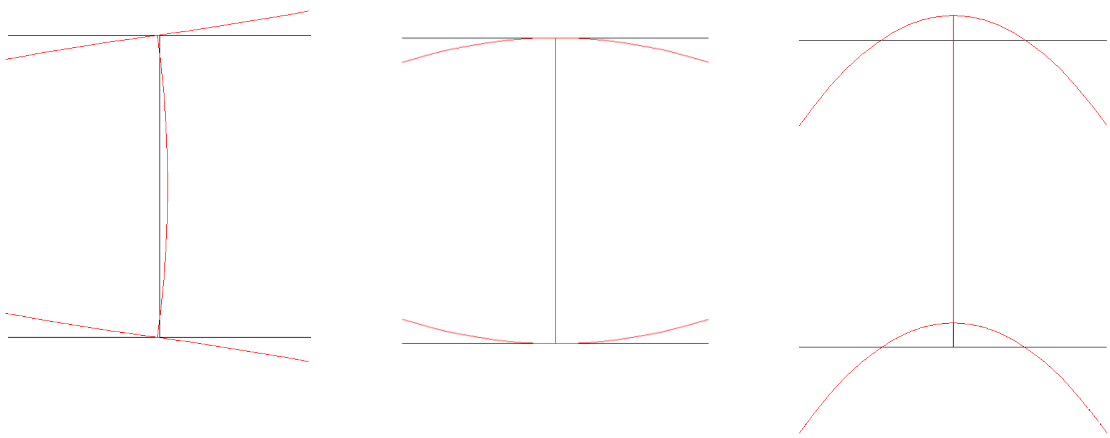


Figure 1: examples of transversal deformation modes for a thin walled profile I-section with 1D elements.

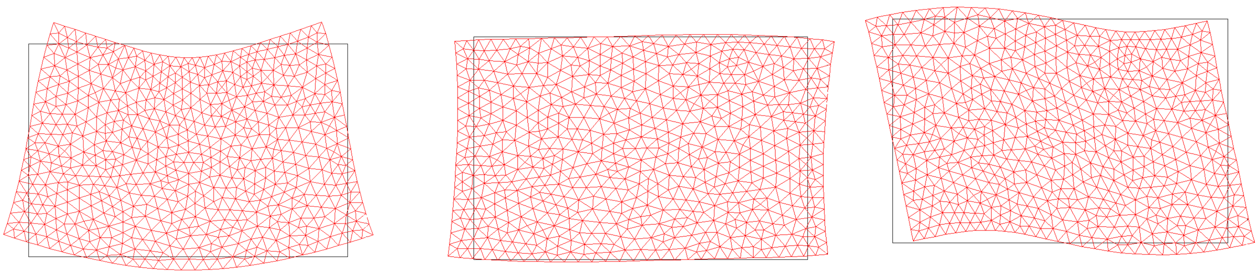


Figure 2: examples of transversal deformation modes for a rectangular section with triangular elements.

In our formulation, the only conditions that needs to be satisfied by the family of transversal modes functions, is that they form a free family, not necessarily orthogonal. Thus, an important feature of our formulation is that any free family can be used to enrich our kinematics, the resolution of the differential equation system, performed later, being completely independent from the choice of the transversal and warping modes functions. In our case, the condition of free family is satisfied by construction, from the solution of the SEP by the well-known Arnoldi iteration algorithm, implemented in ARPACK routines.

In [5] the same method is used to determine the transversal mode for a thin walled profile, with the difference that they use a 3D Timoshenko beam for their section discretization, thus from the resolution of the SEP they derive the transversal modes and also their corresponding warping mode. We use here a different approach for the determination of the 1st warping mode for each transversal mode, based on the equilibrium of the beam element in the case of uniform warping; the higher order warping modes will be derived by using an iterative equilibrium scheme.

3. DETERMINATION OF WARPING FUNCTIONS MODES FOR A GIVEN TRANSVERSAL MODE:

3.1. THE 1ST WARPING MODE DETERMINATION:

We consider the kinematics of a beam element free to warp, where we include only one transversal deformation mode. We then write the displacement vector **d** of an arbitrary point P of the section:

$$\boldsymbol{d} = \begin{Bmatrix} u_p \\ v_p \\ w_p \end{Bmatrix} = \begin{Bmatrix} u_p \\ \zeta\psi_y \\ \zeta\psi_z \end{Bmatrix} \tag{3}$$

u_p Represents the longitudinal displacement due to the warping induced by the transversal mode and $\boldsymbol{\psi} = (\psi_y\ ,\ \psi_z)$ the displacement vector of the transversal deformation mode at the point P.

From the condition of uniform warping in every section (all the beam's cross section will deform in the same manner), we can write:

$$\varepsilon_{xx} = \frac{\partial u_p}{\partial x} = 0 \tag{4}$$

This relation expresses the fact that uniform warping doesn't induce any change in the length of all the beam's fibers.

From the displacement vector, we deduce the strain field:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_p}{\partial x} = 0 & \varepsilon_{yy} &= \frac{\partial \psi_y}{\partial y}\zeta \\ 2\varepsilon_{xy} &= \frac{\partial u_p}{\partial y} + \psi_y \frac{d\zeta}{dx} & \varepsilon_{zz} &= \frac{\partial \psi_z}{\partial z}\zeta \\ 2\varepsilon_{xz} &= \frac{\partial u_p}{\partial z} + \psi_z \frac{d\zeta}{dx} & 2\varepsilon_{yz} &= \zeta\left(\frac{\partial \psi_y}{\partial z} + \frac{\partial \psi_z}{\partial y}\right) \end{aligned} \tag{5}$$

And the stress field:

$$\begin{aligned} \sigma_{xx} &= \lambda\, \zeta\, div\, \boldsymbol{\psi} & \sigma_{yy} &= \left(2\mu \frac{\partial \psi_y}{\partial y} + \lambda\, div\, \boldsymbol{\psi}\right)\zeta \\ \sigma_{xy} &= \mu\left(\frac{\partial u_p}{\partial y} + \psi_y \frac{d\zeta}{dx}\right) & \sigma_{zz} &= \left(2\mu \frac{\partial \psi_z}{\partial z} + \lambda\, div\, \boldsymbol{\psi}\right)\zeta \\ \sigma_{xz} &= \mu\left(\frac{\partial u_p}{\partial z} + \psi_z \frac{d\zeta}{dx}\right) & \sigma_{yz} &= \mu\left(\frac{\partial \psi_y}{\partial z} + \frac{\partial \psi_z}{\partial y}\right)\zeta \end{aligned} \tag{6}$$

Where $\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)}$ and $\mu = \frac{E}{2(1+\upsilon)}$ are the Lamé coefficients, E the elasticity modulus and υ the Poisson coefficient, and div is the divergence operator: $div\, \boldsymbol{\psi} = \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z}$

We note the stress vector: $\boldsymbol{\tau} = (\sigma_{xy}\ , \sigma_{xz}\ )$, that will be expressed by :

$$\boldsymbol{\tau} = \mu\left(\boldsymbol{\nabla} u_p + \frac{d\zeta}{dx}\boldsymbol{\psi}\right) \tag{7}$$

Where $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}\right)$ is the gradient operator.

We write the equilibrium equation of the beam:

$$\frac{\partial \sigma_{xx}}{\partial x} + div\, \boldsymbol{\tau} = 0 \tag{8}$$

$$\lambda\, div\, \boldsymbol{\psi} \frac{d\zeta}{dx} + \mu\left(\Delta u_p + \frac{d\zeta}{dx}\, div\, \boldsymbol{\psi}\right) = 0 \tag{9}$$

$$\Delta u_p = -\left(1 + \frac{\lambda}{\mu}\right) div\, \boldsymbol{\psi}\, \frac{d\zeta}{dx} \tag{10}$$

Where $\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.

From the last equation, we can see that the displacement u_p can be written in the following form $u_p = -\Omega \frac{d\zeta}{dx}$, where Ω is the warping function, verifying the relation :

$$\Delta\Omega = \left(1 + \frac{\lambda}{\mu}\right) div\, \boldsymbol{\psi} \tag{11}$$

The normal component of the stress vector $\boldsymbol{\tau}$ has to be null on the border of the cross section, this condition is expressed by:

$$\boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \tag{12}$$

$$\mu \frac{d\zeta}{dx}(-\boldsymbol{\nabla}\Omega + \boldsymbol{\psi}) \cdot \boldsymbol{n} = 0 \tag{13}$$

$$\frac{\partial \Omega}{\partial n} = \boldsymbol{\psi} \cdot \boldsymbol{n} \tag{14}$$

Where $\boldsymbol{n}$ is the normal vector to the border of the section and $\frac{\partial \Omega}{\partial n} = \boldsymbol{\nabla}\Omega \cdot \boldsymbol{n}$

We resume here the equations of the partial derivatives problem, leading to the determination of the warping function Ω:

$$\begin{aligned} &\Delta\Omega = \left(1 + \frac{\lambda}{\mu}\right) div\, \boldsymbol{\psi} \qquad on\ \ S \\ &\frac{\partial \Omega}{\partial n} = \boldsymbol{\psi} \cdot \boldsymbol{n} \qquad\qquad on\ \Gamma \\ &\int_S \Omega\, dS = 0 \end{aligned} \tag{15}$$

Where S is the cross section area.

The last relation was added to derive Ω uniquely. This relation corresponds to the condition that warping doesn’t induce any uniform displacement.

The resolution of this problem can be performed by using one of the many numerical methods available, such as finite element method (FEM), finite difference method (FDM) or boundary element method (BEM), see [4] for more details.

If we consider the rigid body motion of the beam section, corresponding to the displacements v and w in the two directions of the principle axes and the rotation around the normal axis to the section θ_x, we will obtain the flexural modes and the Vlassov torsion warping [6], by solving the problem above, this shows the analogy between rigid body motion modes and the other transversal deformation modes.

Details about the derivation of the 1st warping mode, for a thin walled profile section, are given in appendix A1.

3.2. DETERMINATION OF HIGHER MODES WARPING FUNCTIONS:

In the case of non-uniform warping, the condition of $\varepsilon_{xx} = 0$ is no longer verified, thus we can’t state that the longitudinal displacement is of the form $u_p = -\Omega \frac{d\zeta}{dx}$, but instead we make the assumption that u_p can be written in the following form :

$$u_p = \Omega\,\delta = \Omega\left(\xi - \frac{d\zeta}{dx}\right) \tag{16}$$

Where δ is the new rate function of the corresponding warping mode.

Thus the new normal stress is:

$$\sigma_{xx} = \underbrace{\lambda\,\zeta\,div\,\boldsymbol{\psi}}_{\sigma^1} + \underbrace{2\mu\,\Omega\,\frac{d\xi}{dx}}_{\sigma^2} \tag{17}$$

The warping function Ω has been calculated in such a way that the normal stress σ^1 is equilibrated and the condition $\boldsymbol{\tau}\cdot\boldsymbol{n} = 0$ verified, thus in the case of a non-uniform warping the total normal stress will not be equilibrated due to the presence of the residual normal stress σ^2. Restoring equilibrium leads to the determination of a secondary shear stress associated to a 2nd warping mode. This reasoning can be considered as the first step of an iterative equilibrium scheme, converging to the exact shape of the warping due to the considered transversal deformation mode.

We assume that we have determined the nth warping mode, and we wish to determine the n+1th warping mode. The nth warping normal stress σ^n will be then equilibrated by the n+1th warping shear stress:

$$\frac{\partial\sigma^n}{\partial x} + div\,\boldsymbol{\tau}^{n+1} = 0 \tag{18}$$

Where : $\sigma^n = 2\mu\,\Omega_n\,\frac{d\xi_n}{dx}$, $\boldsymbol{\tau}^{n+1} = \mu\xi_{n+1}\boldsymbol{\nabla}\Omega_{n+1}$

Thus :

$$2\mu\,\Omega_n\,\frac{d^2\xi_n}{dx^2} + \mu\xi_{n+1}\,\Delta\Omega_{n+1} = 0 \tag{19}$$

The functions Ω_{n+1} and Ω_n depends only of the geometry of the cross section, whereas ξ_{n+1} and ξ_n depends of the abscissa x, so it implies that there exists necessarily two constants α_{n+1} and β_{n+1}, related to the equilibrium of the beam, verifying: $\Delta\Omega_{n+1} = \alpha_{n+1}\Omega_n$, $\xi_{n+1} = \beta_{n+1}\frac{d^2\xi_n}{dx^2}$.

Our goal is to construct a base of warping functions, where any section warping can be decomposed linearly with the aid of the generalized coordinates ξ_i that can be seen as a participation rate for the corresponding warping mode. In practice we need only to determine the warping functions to a multiplicative constant, and the participation rate for each mode will be obtained by writing the equilibrium of the beam. Thus, at the cross section level, only the problem $\Delta\Omega_{n+1} = \Omega_n$ has to be solved. For this problem, we need to define the Neumann and Dirichlet conditions to solve the problem uniquely. The Neumann condition, which specify the value of the derivative of Ω_{n+1} on the border of the section, will have the same form of the 1st warping function ($\frac{\partial \Omega_1}{\partial n} = \boldsymbol{\psi} \cdot \boldsymbol{n}$), in practice we can use the condition $\frac{\partial \Omega_{n+1}}{\partial n} = 0$ to solve the problem, and the uniqueness will be assured by the orthogonalization with the lower modes, to this aim the modified Gram-Schmidt orthogonalization process can be used. For the Dirichlet condition, we impose $\Omega_{n+1} = 0$ at an arbitrary point to solve the problem, and the condition $\int_S \Omega_{n+1} dS = 0$ will assure the uniqueness.

4. EQUILIBRIUM EQUATIONS AND THE STIFFNESS MATRIX:

4.1. KINEMATIC, STRAIN AND STRESS FIELDS :

In this part we will consider general beam kinematics, with n transversal deformation modes and m warping modes, we note that in this section the transversal deformation and warping modes can be completely independent and can be chosen arbitrarily on the unique condition that the family of warping functions and the family of transversal functions should be free. We write the enriched kinematics:

$$\boldsymbol{d}_p = \begin{Bmatrix} u_p \\ v_p \\ w_p \end{Bmatrix} = \begin{Bmatrix} u + \sum_{j=1}^{m} \Omega_j \xi_j \\ \sum_{i=1}^{n} \psi_y^i \zeta^i \\ \sum_{i=1}^{n} \psi_z^i \zeta^i \end{Bmatrix} \tag{20}$$

We note that the first transversal deformation modes will correspond to the rigid body motion, and the first warping modes will correspond to the beam flexion in the direction of the two principal axes. The strain field obtained from this kinematics will be expressed by:

$$\begin{aligned} \varepsilon_{xx} &= \frac{du}{dx} + \Omega_j \frac{d\xi_j}{dx} & \varepsilon_{yy} &= \frac{\partial \psi_y^i}{\partial y} \zeta^i \\ 2\varepsilon_{xy} &= \frac{\partial \Omega_j}{\partial y} \xi_j + \psi_y^i \frac{d\zeta^i}{dx} & \varepsilon_{zz} &= \frac{\partial \psi_z^i}{\partial z} \zeta^i \\ 2\varepsilon_{xz} &= \frac{\partial \Omega_j}{\partial z} \xi_j + \psi_z^i \frac{d\zeta^i}{dx} & 2\varepsilon_{yz} &= \left(\frac{\partial \psi_y^i}{\partial z} + \frac{\partial \psi_z^i}{\partial y} \right) \zeta^i \end{aligned} \tag{21}$$

We have used above and in all that will follow, the Einstein summation convention, to simplify as much as possible the equations expressions.

To determine the stress field from the strain field, we have to make use of the constitutive relation:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & & & \\ & 2\mu+\lambda & \lambda & & 0 & \\ & & 2\mu+\lambda & & & \\ & & & \mu & & \\ & sym & & & \mu & \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix} = \boldsymbol{\Gamma\varepsilon} \tag{22}$$

Thus :

$$\begin{aligned} \sigma_{xx} &= 2\mu\left(\frac{du}{dx} + \Omega_j \frac{d\xi_j}{dx}\right) + \sigma_r & \sigma_{yy} &= 2\mu \frac{\partial \psi_y^i}{\partial y}\zeta^i + \sigma_r \\ \sigma_{xy} &= \mu\left(\frac{\partial \Omega_j}{\partial y}\xi_j + \psi_y^i \frac{d\zeta^i}{dx}\right) & \sigma_{zz} &= 2\mu \frac{\partial \psi_z^i}{\partial z}\zeta^i + \sigma_r \\ \sigma_{xz} &= \mu\left(\frac{\partial \Omega_j}{\partial z}\xi_j + \psi_z^i \frac{d\zeta^i}{dx}\right) & \sigma_{yz} &= \mu\left(\frac{\partial \psi_y^i}{\partial z} + \frac{\partial \psi_z^i}{\partial y}\right)\zeta^i \end{aligned} \tag{23}$$

Where : $\sigma_r = \lambda tr(\boldsymbol{\varepsilon}) = \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) = \lambda\left(\frac{du}{dx} + \zeta^i div\, \boldsymbol{\psi}^i + \Omega_j \frac{d\xi_j}{dx}\right)$.

4.2. THE PRINCIPLE OF VIRTUAL WORK AND THE EQUILIBRIUM EQUATIONS:

We write the expression of the internal virtual work:

$$\delta W_{int} = \int_V \boldsymbol{\sigma}^{\boldsymbol{T}}\, \delta\boldsymbol{\varepsilon}\, dV \tag{24}$$

$$\begin{aligned} \delta W_{int} = \int_V \Bigg(& \sigma_{xx}\left(\frac{d\delta u}{dx} + \Omega_j \frac{d\delta\xi_j}{dx}\right) + \left(\sigma_{xy}\frac{\partial \Omega_j}{\partial y} + \sigma_{xz}\frac{\partial \Omega_j}{\partial z}\right)\delta\xi_j \\ & + \left(\sigma_{xy}\psi_y^i + \sigma_{xz}\psi_z^i\right)\frac{d\delta\zeta^i}{dx} \\ & + \left(\sigma_{yy}\frac{\partial \psi_y^i}{\partial y} + \sigma_{zz}\frac{\partial \psi_z^i}{\partial z} + \sigma_{zy}\left(\frac{\partial \psi_y^i}{\partial z} + \frac{\partial \psi_z^i}{\partial y}\right)\right)\delta\zeta^i \Bigg) dV \end{aligned} \tag{25}$$

We integrate over the whole section to obtain:

$$\delta W_{int} = \int_0^L \left(N\frac{d\delta u}{dx} + M_j \frac{d\delta\xi_j}{dx} + T_j \delta\xi_j + \Lambda^i \frac{d\delta\zeta^i}{dx} + \Phi^i \delta\zeta^i\right) dx \tag{26}$$

The expressions of the generalized efforts are:

$$N = \int_S \sigma_{xx}\, dS = (2\mu + \lambda) S \frac{du}{dx} + \lambda P^i \zeta^i \tag{27}$$

$$M_j = \int_S \Omega_j \sigma_{xx}\, dS = (2\mu + \lambda) K_{jk} \frac{d\xi_k}{dx} + \lambda Q_j^i \zeta^i \tag{28}$$

$$T_j = \int_S \left(\sigma_{xy} \frac{\partial \Omega_j}{\partial y} + \sigma_{xz} \frac{\partial \Omega_j}{\partial z} \right) dS = \mu \left(J_{jk} \xi_k + D_j^l \frac{d\zeta^l}{dx} \right) \tag{29}$$

$$\Lambda^i = \int_S \left(\sigma_{xy} \psi_y^i + \sigma_{xz} \psi_z^i \right) dS = \mu \left(D_k^i \xi_k + C^{il} \frac{d\zeta^l}{dx} \right) \tag{30}$$

$$\Phi^i = \int_S \left(\frac{\partial \psi_z^i}{\partial z} \sigma_{zz} + \frac{\partial \psi_y^i}{\partial y} \sigma_{yy} + \left(\frac{\partial \psi_y^i}{\partial z} + \frac{\partial \psi_z^i}{\partial y} \right) \sigma_{yz} \right) dS = (\mu\, H^{il} + \lambda F^{il}) \zeta^l + \lambda \left(P^i \frac{du}{dx} + Q_j^i \frac{d\xi_j}{dx} \right) \tag{31}$$

The coefficients are expressed by :

$$K_{jk} = \int_S \Omega_j \Omega_k \, dS \quad , \quad D_j^l = \int_S \boldsymbol{\psi}^l \cdot \boldsymbol{\nabla} \Omega_j \, dS \quad , \quad C^{il} = \int_S \boldsymbol{\psi}^l \cdot \boldsymbol{\psi}^i \, dS \quad , \quad J_{jk} = \int_S \boldsymbol{\nabla} \Omega_j \cdot \boldsymbol{\nabla} \Omega_k \, dS$$

$$F^{il} = \int_S div\, \boldsymbol{\psi}^i div\, \boldsymbol{\psi}^l dS \quad , \quad Q_j^i = \int_S \Omega_j \, div\, \boldsymbol{\psi}^i dS \quad , \quad P^i = \int_S div\, \boldsymbol{\psi}^i \, dS \quad , \quad H^{il} = \int_S \Delta(\boldsymbol{\psi}^i, \boldsymbol{\psi}^l)\, dS$$

Where the operator Δ is defined by: $\Delta(\boldsymbol{\psi}^i, \boldsymbol{\psi}^l) = 2\left(\boldsymbol{\nabla} \psi_y^i \cdot \boldsymbol{\nabla} \psi_y^l + \boldsymbol{\nabla} \psi_z^i \cdot \boldsymbol{\nabla} \psi_z^l \right) - curl \boldsymbol{\psi}^i \, curl \boldsymbol{\psi}^l$

And : $curl \boldsymbol{\psi} = \frac{\partial \psi_y}{\partial z} - \frac{\partial \psi_z}{\partial y}$

After integration by parts of the internal virtual work, we obtain:

$$\delta W_{int} = \int_0^L \left(-\frac{dN}{dx} \delta u - \left(\frac{dM_j}{dx} - T_j \right) \delta \xi_j - \left(\frac{d\Lambda^i}{dx} - \Phi^i \right) \delta \zeta^i \right) dx + \underbrace{\left[N \delta u + M_j \delta \xi_j + \Lambda^i \delta \zeta^i \right]_0^L}_{\delta W_{ext}} \tag{32}$$

From the principal of virtual work $\delta W_{int} = \delta W_{ext}$, thus we can write:

$$\int_0^L \left(\frac{dN}{dx} \delta u + \left(\frac{dM_j}{dx} - T_j \right) \delta \xi_j + \left(\frac{d\Lambda^i}{dx} - \Phi^i \right) \delta \zeta^i \right) dx = 0 \tag{33}$$

This relation is valid for any admissible virtual displacements, then the expressions between brackets have to be null:

$$\frac{dN}{dx}=0 \quad , \quad \frac{dM_j}{dx}-T_j=0 \quad for \quad 1\le j\le m \quad , \quad \frac{d\Lambda^i}{dx}-\Phi^i=0 \quad for \quad 1\le i\le n \tag{34}$$

We develop the two last equations:

$$\begin{cases} (2\mu+\lambda)K_{jk}\dfrac{d^2\xi_k}{dx^2}+\lambda Q_j^l\dfrac{d\zeta^l}{dx}-\mu\left(J_{jk}\xi_k+D_j^l\dfrac{d\zeta^l}{dx}\right)=0 \\ \mu\left(D_k^i\dfrac{d\xi_k}{dx}+C^{il}\dfrac{d^2\zeta^l}{dx^2}\right)-(\mu\, H^{il}+\lambda F^{il})\zeta^l-\lambda\left(P^i\dfrac{du}{dx}+Q_k^i\dfrac{d\xi_k}{dx}\right)=0 \end{cases} \tag{35}$$

$$\Rightarrow \begin{cases} (2\mu+\lambda)K_{jk}\dfrac{d^2\xi_k}{dx^2}-\mu B_j^l\dfrac{d\zeta^l}{dx}-\mu J_{jk}\xi_k=0 \\ \mu C^{il}\dfrac{d^2\zeta^l}{dx^2}+\mu B_k^i\dfrac{d\xi_k}{dx}-(\mu\, H^{il}+\lambda F^{il})\,\zeta^l=\lambda P^i\dfrac{du}{dx} \end{cases} \tag{36}$$

From the expression of the normal force, we have the following relation:

$$\frac{du}{dx}=\frac{1}{(2\mu+\lambda)S}(N-\lambda P^l\zeta^l) \tag{37}$$

Thus, if we replace it in the 2nd equation in (36) we obtain :

$$\mu C^{il}\frac{d^2\zeta^l}{dx^2}+\mu B_k^i\frac{d\xi_k}{dx}-\left(\mu\, H^{il}+\lambda F^{il}-\frac{\lambda^2P^iP^l}{(2\mu+\lambda)S}\right)\zeta^l=\frac{\lambda P^i}{(2\mu+\lambda)S}N \tag{38}$$

We introduce some notations:

$$\boldsymbol{K}=(2\mu+\lambda)\int_S\begin{bmatrix}\Omega_1\Omega_1 & \dots & \Omega_1\Omega_m\\ & \ddots & \vdots\\ sym & & \Omega_m\Omega_m\end{bmatrix}dS \quad , \quad \boldsymbol{C}=\mu\int_S\begin{bmatrix}\boldsymbol{\psi}^1\cdot\boldsymbol{\psi}^1 & \dots & \boldsymbol{\psi}^1\cdot\boldsymbol{\psi}^n\\ & \ddots & \vdots\\ sym & & \boldsymbol{\psi}^n\cdot\boldsymbol{\psi}^n\end{bmatrix}dS$$

$$\boldsymbol{J}=\mu\int_S\begin{bmatrix}\boldsymbol{\nabla}\Omega_1\cdot\boldsymbol{\nabla}\Omega_m & \dots & \boldsymbol{\nabla}\Omega_1\cdot\boldsymbol{\nabla}\Omega_m\\ & \ddots & \vdots\\ sym & & \boldsymbol{\nabla}\Omega_m\cdot\boldsymbol{\nabla}\Omega_m\end{bmatrix}dS \quad , \quad \boldsymbol{p}=\frac{\lambda}{(2\mu+\lambda)S}\begin{Bmatrix}P^1\\ \vdots\\ P^n\end{Bmatrix}$$

$$\boldsymbol{D}=\mu\int_S\begin{bmatrix}\boldsymbol{\psi}^1\cdot\boldsymbol{\nabla}\Omega_1 & \dots & \boldsymbol{\psi}^n\cdot\boldsymbol{\nabla}\Omega_1\\ \vdots & \ddots & \vdots\\ \boldsymbol{\psi}^1\cdot\boldsymbol{\nabla}\Omega_m & \dots & \boldsymbol{\psi}^n\cdot\boldsymbol{\nabla}\Omega_m\end{bmatrix}dS \quad , \quad \boldsymbol{Q}=\lambda\int_S\begin{bmatrix}\Omega_1\, div\,\boldsymbol{\psi}^1 & \dots & \Omega_1\, div\,\boldsymbol{\psi}^n\\ \vdots & \ddots & \vdots\\ \Omega_m\, div\,\boldsymbol{\psi}^1 & \dots & \Omega_m\, div\,\boldsymbol{\psi}^n\end{bmatrix}dS$$

$$\boldsymbol{F}=\lambda\int_S\begin{bmatrix}div\,\boldsymbol{\psi}^1 div\,\boldsymbol{\psi}^1 & \dots & div\,\boldsymbol{\psi}^1 div\,\boldsymbol{\psi}^n\\ & \ddots & \vdots\\ sym & & div\,\boldsymbol{\psi}^n div\,\boldsymbol{\psi}^n\end{bmatrix}dS \quad , \quad \boldsymbol{H}=\mu\int_S\begin{bmatrix}\varDelta(\boldsymbol{\psi}^1,\boldsymbol{\psi}^1) & \dots & \varDelta(\boldsymbol{\psi}^1,\boldsymbol{\psi}^n)\\ & \ddots & \vdots\\ sym & & \varDelta(\boldsymbol{\psi}^n,\boldsymbol{\psi}^n)\end{bmatrix}dS$$

And: $\boldsymbol{A}=\boldsymbol{H}+\boldsymbol{F}-(2\mu+\lambda)S\,\boldsymbol{p}\otimes\boldsymbol{p} \quad , \quad \boldsymbol{B}=\boldsymbol{D}-\boldsymbol{Q}$

The system of differential equations can be written now in the following matrix form :

$$\boldsymbol{K}\boldsymbol{\xi}'' - \boldsymbol{B}\boldsymbol{\zeta}' - \boldsymbol{J}\boldsymbol{\xi} = 0 \tag{39}$$

$$\boldsymbol{C}\boldsymbol{\zeta}'' + \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{\xi}' - \boldsymbol{A}\boldsymbol{\zeta} = \boldsymbol{p}N \tag{40}$$

Where : $\boldsymbol{\xi} = \begin{Bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{Bmatrix}$, $\boldsymbol{\zeta} = \begin{Bmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{Bmatrix}$

We note that the matrix $\boldsymbol{C}$ will be diagonal, but not necessarily the matrix $\boldsymbol{K}$, since the warping functions from different transversal modes are not necessarily orthogonal. We note also that the differential equation system obtained in (39) and (40), is analogical to a dynamical equilibrium equation system for a gyroscopic system in an unstable state. In the next part, we will solve this system exactly to assemble the rigidity matrix; the general solution will prove to be a little exhaustive because we need to separate in our system the rigid body motion and flexural modes from the others, and this is done by making two variable changes, one for the transversal modes and another for the warping modes.

4.3. RESOLUTION OF THE EQUILIBRIUM EQUATIONS AND DERIVATION OF THE STIFFNESS MATRIX:

In order to solve the differential equation system written in (39) and (40), we first need to extract the rigid body motion modes, to this aim we will start by solving the generalized eigenvalue problem (GEP) $\boldsymbol{Az} = \alpha\boldsymbol{Cz}$. The matrix $\boldsymbol{C}$ is the gramian matrix attached to the basis formed by the transversal deformation mode vectors, it will be then a definite positive matrix, and by noticing that $\boldsymbol{A}$ is a symmetric matrix, we can say then that the eigenvalues of our problem will be positive $\alpha \geq 0$, and the number of zero eigenvalues will in fact be equal to the number of rigid body motion modes, that can be superior to three for some cases, for example for a global cross section formed by n disjoints sections, then the number of rigid body modes will be equal to $3n$.

After solving the GEP, we obtain the eigenvectors matrix $\boldsymbol{P}$ verifying:

$$\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{0}_{q\times q} & \\ & \boldsymbol{G} \end{bmatrix} \quad , \quad \boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P} = \boldsymbol{I} \tag{41}$$

Where $\boldsymbol{G}$ is the diagonal matrix containing the non-zero eigenvalues and q the number of rigid body modes.

We proceed with a variable change in the eigenvectors base: $\boldsymbol{\zeta} = \boldsymbol{P}\boldsymbol{\vartheta}_t = \boldsymbol{P}\begin{Bmatrix}\boldsymbol{\vartheta}_q \\ \boldsymbol{\vartheta}\end{Bmatrix}$

And we note: $\boldsymbol{P} = [\boldsymbol{P}_1 \quad \boldsymbol{P}_2]$, $\boldsymbol{B}_i = \boldsymbol{B}\,\boldsymbol{P}_i$, $\boldsymbol{p}_i = \boldsymbol{P}_i^T\,\boldsymbol{p} \quad for \;\; i = 1{,}2$

We substitute this in equation (40) to obtain:

$$\boldsymbol{P}^T\boldsymbol{C}\boldsymbol{P}\boldsymbol{\vartheta}_t{}'' + \boldsymbol{P}^T\boldsymbol{B}^T\boldsymbol{\xi}' - \boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}\boldsymbol{\vartheta}_t = \boldsymbol{P}^T\boldsymbol{p}N$$

$$\boldsymbol{\vartheta}_q{}'' + \boldsymbol{B}_1^T\,\boldsymbol{\xi}' = \boldsymbol{p}_1 N \tag{42}$$

$$\boldsymbol{\vartheta}'' - \boldsymbol{G}\boldsymbol{\vartheta} + \boldsymbol{B}_2^T\,\boldsymbol{\xi}' = \boldsymbol{p}_2 N \tag{43}$$

We make also the variable change into the equation (39) to fully transform our system with the new variables:

$$\boldsymbol{K}\boldsymbol{\xi}'' - \boldsymbol{J}\boldsymbol{\xi} - \boldsymbol{B}_1\boldsymbol{\vartheta}_q{}' - \boldsymbol{B}_2\,\boldsymbol{\vartheta}' = \boldsymbol{0} \tag{44}$$

We integrate the equation (42) between 0 and x:

$$\boldsymbol{\vartheta}_q{}' = \boldsymbol{\vartheta}_{q0}{}' - \boldsymbol{B}_1^T\,(\boldsymbol{\xi} - \boldsymbol{\xi}_0) + \boldsymbol{p}_1 N x \tag{45}$$

We also transform the generalized effort vector $\boldsymbol{\gamma} = \{\Lambda^i\}_{1\le i\le n}$ associated to $\boldsymbol{\zeta}$:

$$\boldsymbol{\Gamma}_t = \begin{Bmatrix}\boldsymbol{\Gamma}_q \\ \boldsymbol{\Gamma}\end{Bmatrix} = \boldsymbol{P}^T\boldsymbol{\gamma} = \boldsymbol{\vartheta}'_t + \boldsymbol{P}^T\boldsymbol{D}^T\boldsymbol{\xi} \tag{46}$$

$$\Rightarrow \quad \begin{cases}\boldsymbol{\vartheta}_{q0}{}' = \boldsymbol{\Gamma}_{q0} - \boldsymbol{D}_1^T\boldsymbol{\xi}_0 \\ \boldsymbol{\Gamma} = \boldsymbol{\vartheta}' + \boldsymbol{D}_2^T\boldsymbol{\xi}\end{cases} \quad \Rightarrow \quad \begin{cases}\boldsymbol{\vartheta}_q{}' = \boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_1^T\,\boldsymbol{\xi}_0 + \boldsymbol{p}_1 N x - \boldsymbol{B}_1^T\boldsymbol{\xi} \\ \boldsymbol{\Gamma} = \boldsymbol{\vartheta}' + \boldsymbol{D}_2^T\boldsymbol{\xi}\end{cases} \tag{47}$$

Where: $\boldsymbol{Q}_1 = \boldsymbol{Q}\,\boldsymbol{P}_1 = (\boldsymbol{D} - \boldsymbol{B})\boldsymbol{P}_1 = \boldsymbol{D}_1 - \boldsymbol{B}_1$.

If we substitute the expression of $\boldsymbol{\vartheta}_q{}'$ obtained in (47) into the equation (44) we obtain:

$$\boldsymbol{K}\boldsymbol{\xi}'' - (\boldsymbol{J} - \boldsymbol{B}_1\boldsymbol{B}_1^T)\boldsymbol{\xi} - \boldsymbol{B}_2\boldsymbol{\vartheta}' = \boldsymbol{B}_1\big(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_1^T\,\boldsymbol{\xi}_0 + \boldsymbol{p}_1 N x\big) \tag{48}$$

Thus, the system to solve is transformed into a new equivalent one:

$$\boldsymbol{K}\boldsymbol{\xi}'' - \boldsymbol{B}_2\boldsymbol{\vartheta}' - \boldsymbol{J}_q\boldsymbol{\xi} = \boldsymbol{B}_1\big(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_1^T\,\boldsymbol{\xi}_0 + \boldsymbol{p}_1 N x\big) \tag{49}$$

$$\boldsymbol{\vartheta}'' + \boldsymbol{B}_2^T\,\boldsymbol{\xi}' - \boldsymbol{G}\boldsymbol{\vartheta} = \boldsymbol{p}_2 N \tag{50}$$

Where : $\boldsymbol{J}_q = \boldsymbol{J} - \boldsymbol{B}_1\boldsymbol{B}_1^T$

As for the matrix $\boldsymbol{A}$, the symmetric matrix $\boldsymbol{J}_q$ will contain some zero eigenvalues that correspond this time to the flexural modes, and in order to solve our system we also need to extract these modes and separate them from the other warping modes, thus we have to solve the GEP $\boldsymbol{J}_q\boldsymbol{z} = \alpha\boldsymbol{K}\boldsymbol{z}$, with $\boldsymbol{K}$ the gramian matrix attached to the warping functions base, thus it's a definite positive matrix, and as previously stated, this means that $\alpha \geq 0$.

After solving the eigenvalue problem, we obtain the eigenvectors matrix $\boldsymbol{R}$ verifying:

$$\boldsymbol{R}^T\boldsymbol{J}_q\boldsymbol{R} = \begin{bmatrix} \boldsymbol{0}_{l\times l} & \\ & \boldsymbol{S} \end{bmatrix} \quad , \quad \boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R} = \boldsymbol{I} \tag{51}$$

Where $\boldsymbol{S}$ is the diagonal matrix containing the non-zero eigenvalues and l the number of the flexural modes, that will be equal in general to $l = 2q/3$.

We proceed to a second variable change: $\boldsymbol{\xi} = \boldsymbol{R}\boldsymbol{\varphi}_t = \boldsymbol{R}\begin{Bmatrix}\boldsymbol{\varphi}_l\\ \boldsymbol{\varphi}\end{Bmatrix}$

And we note: $\boldsymbol{R} = [\boldsymbol{R}_1 \quad \boldsymbol{R}_2] \quad , \quad \boldsymbol{B}_{ij} = \boldsymbol{R}_j^T\boldsymbol{B}_i \quad , \quad \boldsymbol{Q}_{ij} = \boldsymbol{R}_j^T\boldsymbol{Q}_i \quad for\ i = 1{,}2\ and\ j = 1{,}2$

We replace in the equation (49) to obtain:

$$\boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R}\boldsymbol{\varphi}_t{}'' - \boldsymbol{R}^T\boldsymbol{J}_q\boldsymbol{R}\boldsymbol{\varphi}_t - \boldsymbol{R}^T\boldsymbol{B}_2\boldsymbol{\vartheta}' = \boldsymbol{R}^T\boldsymbol{B}_1\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_1^T\,\boldsymbol{\xi}_0 + \boldsymbol{p}_1 N x\right) \tag{52}$$

$$\boldsymbol{\varphi}_l{}'' - \boldsymbol{B}_{21}\boldsymbol{\vartheta}' = \boldsymbol{B}_{11}\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_1^T\,\boldsymbol{\xi}_0 + \boldsymbol{p}_1 N x\right) \tag{53}$$

$$\boldsymbol{\varphi}'' - \boldsymbol{S}\boldsymbol{\varphi} - \boldsymbol{B}_{22}\,\boldsymbol{\vartheta}' = \boldsymbol{B}_{12}\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_1^T\,\boldsymbol{\xi}_0 + \boldsymbol{p}_1 N x\right) \tag{54}$$

We integrate the equation (53) from 0 to x:

$$\boldsymbol{\varphi}_l{}' = \boldsymbol{B}_{21}\boldsymbol{\vartheta} + \boldsymbol{h}_x \tag{55}$$

Where: $\boldsymbol{h}_x = \boldsymbol{\varphi}_{l0}{}' - \boldsymbol{B}_{21}\boldsymbol{\vartheta}_0 + \boldsymbol{B}_{11}\left(\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right)x + \boldsymbol{p}_1 N\frac{x^2}{2}\right)$.

By replacing (55) in the equation (50) after making the variable change, we have:

$$\boldsymbol{\vartheta}'' - \boldsymbol{G}\boldsymbol{\vartheta} + \boldsymbol{B}_{22}^T\boldsymbol{\varphi}' + \boldsymbol{B}_{21}^T\,\boldsymbol{\varphi}_l{}' = \boldsymbol{p}_2 N \tag{56}$$

$$\boldsymbol{\vartheta}'' - (\boldsymbol{G} - \boldsymbol{B}_{21}^T\boldsymbol{B}_{21})\boldsymbol{\vartheta} + \boldsymbol{B}_{22}^T\boldsymbol{\varphi}' = \boldsymbol{p}_2 N - \boldsymbol{B}_{21}^T\,\boldsymbol{h}_x \tag{57}$$

We transform the generalized effort vector $\boldsymbol{m} = \{M_j\}_{1\leq j\leq m}$ associated to $\boldsymbol{\xi}$:

$$\boldsymbol{\Upsilon}_t = \begin{Bmatrix}\boldsymbol{\Upsilon}_l\\ \boldsymbol{\Upsilon}\end{Bmatrix} = \boldsymbol{R}^T\boldsymbol{m} = \boldsymbol{R}^T(\boldsymbol{K}\boldsymbol{\xi}' + \boldsymbol{Q}\boldsymbol{\zeta}) = \boldsymbol{\varphi}'_t + \boldsymbol{R}^T\boldsymbol{Q}\boldsymbol{\zeta} = \boldsymbol{\varphi}'_t + \boldsymbol{R}^T\boldsymbol{Q}_1\boldsymbol{\vartheta}_q + \boldsymbol{R}^T\boldsymbol{Q}_2\boldsymbol{\vartheta} \tag{58}$$

$$\Rightarrow \begin{cases}\boldsymbol{\varphi}_{l0}{}' = \boldsymbol{\Upsilon}_{l0} - \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_{q0} - \boldsymbol{Q}_{21}\boldsymbol{\vartheta}_0\\ \boldsymbol{\Upsilon} = \boldsymbol{\varphi}' + \boldsymbol{Q}_{12}\boldsymbol{\vartheta}_q + \boldsymbol{Q}_{22}\boldsymbol{\vartheta}\end{cases} \Rightarrow \begin{cases}\boldsymbol{\varphi}_l{}' = \boldsymbol{B}_{21}\boldsymbol{\vartheta} + \boldsymbol{h}_x\\ \boldsymbol{\Upsilon} = \boldsymbol{\varphi}' + \boldsymbol{Q}_{12}\boldsymbol{\vartheta}_q + \boldsymbol{Q}_{22}\boldsymbol{\vartheta}\end{cases} \tag{59}$$

Where $\boldsymbol{h}_x$ becomes : $\boldsymbol{h}_x = \boldsymbol{\Upsilon}_{l0} - \boldsymbol{D}_{21}\boldsymbol{\vartheta}_0 - \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_{q0} + \boldsymbol{B}_{11}\left(\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right)x + \boldsymbol{p}_1 N\frac{x^2}{2}\right)$.

We note:

$$\boldsymbol{G}_* = \boldsymbol{G} - \boldsymbol{B}_{21}^T\boldsymbol{B}_{21} \ , \ \boldsymbol{f}_x = \begin{Bmatrix}\boldsymbol{f}_1\\ \boldsymbol{f}_2\end{Bmatrix} = \begin{Bmatrix}\boldsymbol{B}_{12}\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0 + \boldsymbol{p}_1 N x\right)\\ \boldsymbol{p}_2 N - \boldsymbol{B}_{21}^T\,\boldsymbol{h}_x\end{Bmatrix}$$

$$\boldsymbol{\phi} = \begin{Bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\vartheta} \end{Bmatrix} \quad , \quad \boldsymbol{T} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \quad , \quad \boldsymbol{U} = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{B}_{22} \\ \boldsymbol{B}_{22}^T & \boldsymbol{0} \end{bmatrix} \quad , \quad \boldsymbol{V} = \begin{bmatrix} \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_* \end{bmatrix}$$

Thus, from (54) and (57) we can write the final system to solve:

$$\begin{cases} \boldsymbol{\varphi}'' - \boldsymbol{B}_{22}\,\boldsymbol{\vartheta}' - \boldsymbol{S}\boldsymbol{\varphi} = \boldsymbol{f}_1 \\ \boldsymbol{\vartheta}'' + \boldsymbol{B}_{22}^T\boldsymbol{\varphi}' - \boldsymbol{G}_*\boldsymbol{\vartheta} = \boldsymbol{f}_2 \end{cases} \quad \Longrightarrow \quad \boldsymbol{T}\boldsymbol{\phi}'' + \boldsymbol{U}\boldsymbol{\phi}' - \boldsymbol{V}\boldsymbol{\phi} = \boldsymbol{f}_x \tag{60}$$

We resume here all the equations that form our new system, that we will solve exactly, equivalent to the initial system in (39) and (40):

$$\begin{cases} u' = \dfrac{N}{(2\mu+\lambda)S} - \boldsymbol{p}_1\cdot\boldsymbol{\vartheta}_q - \boldsymbol{p}_2\cdot\boldsymbol{\vartheta} \\ \boldsymbol{\vartheta}_q{}' = \boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\,\boldsymbol{\varphi}_0 - \boldsymbol{B}_{11}^T\boldsymbol{\varphi}_l - \boldsymbol{B}_{12}^T\boldsymbol{\varphi} + \boldsymbol{p}_1 N x \\ \boldsymbol{\varphi}_l{}' = \boldsymbol{B}_{21}\boldsymbol{\vartheta} + \boldsymbol{h}_x \\ \boldsymbol{T}\boldsymbol{\phi}'' + \boldsymbol{U}\boldsymbol{\phi}' - \boldsymbol{V}\boldsymbol{\phi} = \boldsymbol{f}_x \end{cases} \tag{61}$$

We note $k = m + n - q - l$ the dimension of the last 2nd order differential equation system.

Now that we have uncoupled the rigid body motion and flexural modes from the other modes, we detail the solution of the differential equation system (62) without the second member:

$$\boldsymbol{T}\boldsymbol{\phi}'' + \boldsymbol{U}\boldsymbol{\phi}' - \boldsymbol{V}\boldsymbol{\phi} = \boldsymbol{0} \tag{62}$$

$$\boldsymbol{T}\partial_x^2\boldsymbol{\phi} + \boldsymbol{U}\partial_x\boldsymbol{\phi} - \boldsymbol{V}\boldsymbol{\phi} = \boldsymbol{0} \quad \Longrightarrow \quad (\boldsymbol{T}\partial_x^2 + \boldsymbol{U}\partial_x - \boldsymbol{V})\boldsymbol{\phi} = \boldsymbol{0} \tag{63}$$

Where ∂_x is the derivate about x operator.

To solve the system we will need then to solve its characteristic equation:

$$\det(\boldsymbol{T}\omega^2 + \boldsymbol{U}\omega - \boldsymbol{V}) = \det(\boldsymbol{T})\omega^{2k} + lower\ order\ term = 0 \tag{64}$$

T is a positive definite matrix, thus det(***T***)>0, this implies that the roots of the equation (64) with their multiplicity will be equal to 2k. Solving this equation corresponds in fact to solving a special class of eigenvalue problem called the quadratic eigenvalue problem QEP, see[7], where we need to find $(\omega, \boldsymbol{z}) \in \mathbb{C}^{k+1}$ verifying:

$$\boldsymbol{G}(\omega) = (\boldsymbol{T}\omega^2 + \boldsymbol{U}\omega - \boldsymbol{V})\boldsymbol{z} = 0$$

Proposition 1: If $\boldsymbol{T}$ and $\boldsymbol{V}$ are definite positive matrices, then the eigenvalues of the QEP $\boldsymbol{G}(\omega)\boldsymbol{z} = 0$ will be finite and non-null.

Proof: We introduce some notations : $t(\boldsymbol{z}) = \boldsymbol{z}^*\boldsymbol{T}\boldsymbol{z}$, $u(\boldsymbol{z}) = \boldsymbol{z}^*\boldsymbol{U}\boldsymbol{z}$, $v(\boldsymbol{z}) = \boldsymbol{z}^*\boldsymbol{V}\boldsymbol{z}$, where * define the conjugate transpose of a matrix. For $(\omega, \boldsymbol{z})$ an Eigen-pair, we can write:

$$\boldsymbol{G}(\omega)\boldsymbol{z} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{z}^*\boldsymbol{G}(\omega)\boldsymbol{z} = t(\boldsymbol{z})\omega^2 + u(\boldsymbol{z})\omega - v(\boldsymbol{z}) = 0 \tag{65}$$

The solution of the 2nd order equation $\boldsymbol{z}^*\boldsymbol{G}(\omega)\boldsymbol{z} = 0$ will be written in the following form:

$$\omega = \frac{-u(\boldsymbol{z}) \pm \sqrt{u(\boldsymbol{z})^2 + 4t(\boldsymbol{z})v(\boldsymbol{z})}}{2t(\boldsymbol{z})} \tag{66}$$

$\boldsymbol{T}$ is a definite positive matrix thus $t(\boldsymbol{z}) > 0$, it implies that ω will have finite real values. $\boldsymbol{V}$ is also a definite positive matrix then $t(\boldsymbol{z})v(\boldsymbol{z}) > 0$, thus $\sqrt{u(\boldsymbol{z})^2 + 4t(\boldsymbol{z})v(\boldsymbol{z})} \neq |u(\boldsymbol{z})| \;\Rightarrow\; \omega \neq 0$.

■

Proposition 2: For $\boldsymbol{T}$ and $\boldsymbol{U}$ real symmetric matrices and $\boldsymbol{V}$ a real skew-symmetric matrix, then the eigenvalues of the QEP will have Hamiltonian properties, which mean that they are symmetric about the real and imaginary axis of the complex plane.

Proof: $\boldsymbol{G}$ verifies the following relations:

$$\boldsymbol{G}(\omega)^* = \overline{\boldsymbol{G}(\omega)^{\boldsymbol{T}}} = \boldsymbol{T}\overline{\omega}^2 - \boldsymbol{U}\overline{\omega} - \boldsymbol{V} = \boldsymbol{G}(-\overline{\omega}) \tag{67}$$

$\boldsymbol{T}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ are real matrices, thus : $\boldsymbol{G}(\omega)^T = \boldsymbol{G}(-\omega)$

From these two relations verified by $\boldsymbol{G}$ we can deduce that if ω is an eigenvalue then $\overline{\omega}$, $-\overline{\omega}$ and $-\omega$ are also eigenvalues of the problem.

■

The solution of a QEP is performed with the aid of a linearization, which transforms the problem to a classical generalized eigenvalue problem:

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{V} & -\boldsymbol{U} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w} \\ \boldsymbol{y} \end{Bmatrix} = \omega \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w} \\ \boldsymbol{y} \end{Bmatrix} \tag{68}$$

Where : $\boldsymbol{w} = \boldsymbol{z},\; \boldsymbol{y} = \omega\boldsymbol{z}$

After the resolution of this problem we obtain *2k* eigenvalues assembled in the diagonal matrix $\boldsymbol{S}$, and $\boldsymbol{R}_{k\times 2k}$ there corresponding eigenvectors matrix. Thus the solution of the system can be written in the following form:

$$\boldsymbol{\phi}_x = \boldsymbol{R}e^{\boldsymbol{S}x}\boldsymbol{a} \tag{69}$$

Where $\boldsymbol{a} \in \mathbb{R}^{2k}$ is a vector of arbitrary constants that will be expressed later in function of the boundary conditions.

Knowing that the eigenvalues verify Hamiltonian properties, we can re-arrange the matrix S and R in the following way:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 & & & \mathbf{0} \\ & -\boldsymbol{S}_1 & & \\ & & \boldsymbol{S}_2 & \\ \mathbf{0} & & & \overline{\boldsymbol{S}_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_1 & & & \mathbf{0} \\ & -\boldsymbol{S}_1 & & \\ & & \boldsymbol{S}_{2r} & \\ \mathbf{0} & & & \boldsymbol{S}_{2r} \end{bmatrix} + i \begin{bmatrix} \mathbf{0} & & & \mathbf{0} \\ & \mathbf{0} & & \\ & & \boldsymbol{S}_{2i} & \\ \mathbf{0} & & & -\boldsymbol{S}_{2i} \end{bmatrix} = \boldsymbol{S}_r + i\boldsymbol{S}_i \tag{70}$$

$$\boldsymbol{R} = [\ \boldsymbol{\Phi}_1 \quad -\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2 \quad \overline{\boldsymbol{\Phi}}_2\ ] = \boldsymbol{R}_r + i\boldsymbol{R}_i \tag{71}$$

Where $\boldsymbol{S}_r$, $\boldsymbol{S}_i$, $\boldsymbol{R}_r$ and $\boldsymbol{R}_i$ are real matrices, and i is the complex number verifying i^2=-1.

Proposition 3 : The solution of the differential equation system in (62) can be written in a real form as follows:

$$\boldsymbol{\phi}_x = (\boldsymbol{R}_r\boldsymbol{Y}_x + \boldsymbol{R}_i\boldsymbol{Z}_x)\boldsymbol{e}^{\boldsymbol{S}_r x}\boldsymbol{a} \tag{72}$$

Where:

$$\boldsymbol{Y}_x = \begin{bmatrix} \boldsymbol{I} & & & \mathbf{0} \\ & \boldsymbol{I} & & \\ & & \boldsymbol{X}_{sx} & \\ \mathbf{0} & & & \boldsymbol{X}_{cx} \end{bmatrix}, \quad \boldsymbol{Z}_x = \begin{bmatrix} \boldsymbol{I} & & & \mathbf{0} \\ & \boldsymbol{I} & & \\ & & \boldsymbol{X}_{cx} & \\ \mathbf{0} & & & \boldsymbol{X}_{sx} \end{bmatrix}, \quad \boldsymbol{X}_{cx} = cos(\boldsymbol{S}_{2i}x) \ , \ \boldsymbol{X}_{sx} = sin(\boldsymbol{S}_{2i}x) \tag{73}$$

Proof : see appendix A3.

■

To complete the solution of our system, we write its particular solution:

$$\boldsymbol{\phi}_{px} = \int_0^x \boldsymbol{R}\boldsymbol{e}^{\boldsymbol{S}(x-t)}\boldsymbol{L}\boldsymbol{f}_t\, dt = \boldsymbol{R}\boldsymbol{e}^{\boldsymbol{S}x}\int_0^x \boldsymbol{e}^{-\boldsymbol{S}t}\boldsymbol{L}\boldsymbol{f}_t\, dt \tag{74}$$

Where $\boldsymbol{R}$ and $\boldsymbol{L}$ are respectively the right and left eigenvectors, verifying the following relations:

$$\boldsymbol{RL} = \mathbf{0} \quad , \quad \boldsymbol{TRSL} = \boldsymbol{I} \tag{75}$$

And :

$$\boldsymbol{f}_x = \begin{Bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{B}_{12}\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0 + \boldsymbol{p}_1 N x\right) \\ \boldsymbol{p}_2 N - \boldsymbol{B}_{21}^T\left(\boldsymbol{\Upsilon}_{l0} - \boldsymbol{D}_{21}\boldsymbol{\vartheta}_0 - \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_{q0} + \boldsymbol{B}_{11}\left(\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right)x + \boldsymbol{p}_1 N\frac{x^2}{2}\right)\right) \end{Bmatrix} \tag{76}$$

$$\boldsymbol{f}_x = \underbrace{\begin{Bmatrix} \boldsymbol{B}_{12}\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right) \\ \boldsymbol{p}_2 N - \boldsymbol{B}_{21}^T\left(\boldsymbol{\Upsilon}_{l0} - \boldsymbol{D}_{21}\boldsymbol{\vartheta}_0 - \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_{q0}\right)\end{Bmatrix}}_{\boldsymbol{g}_1} + \underbrace{\begin{Bmatrix} \boldsymbol{B}_{12}\boldsymbol{p}_1 N \\ -\boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right)\end{Bmatrix}}_{\boldsymbol{g}_2} x - \underbrace{\frac{1}{2}\begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\boldsymbol{p}_1 N\end{Bmatrix}}_{\boldsymbol{g}_3} x^2 \quad (77)$$

$\boldsymbol{f}_x = \boldsymbol{g}_1 + \boldsymbol{g}_2 x - \boldsymbol{g}_3 x^2$

Proposition 4 : For a differential equation system: $\boldsymbol{T\phi}'' + \boldsymbol{U\phi}' - \boldsymbol{V\phi} = \boldsymbol{f}_x$, if the second member $\boldsymbol{f}_x$ has the following form: $\boldsymbol{f}_x = \sum_{i=0}^{n} \boldsymbol{f}_i x^i$, then the particular solution $\boldsymbol{\phi}_p$ of the system can be written as follows:

$$\boldsymbol{\phi}_p = -\sum_{i=0}^{n} \boldsymbol{R}\boldsymbol{S}^{-i-1}\boldsymbol{L}\,\boldsymbol{f}_x{}^{(i)} \quad (78)$$

Where $\boldsymbol{f}_x{}^{(i)}$ denote the ith derivate of $\boldsymbol{f}_x$ about x.

Proof: see appendix A4.

■

Thus, the particular solution of our system is:

$$\boldsymbol{\phi}_{px} = -\boldsymbol{V}^{-1}\boldsymbol{f}_x - \boldsymbol{V}^{-1}\boldsymbol{U}\boldsymbol{V}^{-1}(\boldsymbol{g}_2 - 2\boldsymbol{g}_3 x) + 2(\boldsymbol{V}^{-1}\boldsymbol{T}\boldsymbol{V}^{-1} + \boldsymbol{V}^{-1}(\boldsymbol{U}\boldsymbol{V}^{-1})^2)\boldsymbol{g}_3 \quad (79)$$

We can write the total solution of the system :

$$\boldsymbol{\phi}_x = \boldsymbol{W}_x\boldsymbol{a} + \boldsymbol{G}_1\boldsymbol{\vartheta}_{q0} + \boldsymbol{G}_2\boldsymbol{\vartheta}_0 + (\boldsymbol{G}_3 + \boldsymbol{G}_4 x)\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right) + \boldsymbol{G}_5\boldsymbol{\Upsilon}_{l0} + (\boldsymbol{G}_6 + \boldsymbol{G}_7 x + \boldsymbol{G}_8 x^2)N \quad (80)$$

Where :

$\boldsymbol{W}_x = (\boldsymbol{R}_r\boldsymbol{Y}_x + \boldsymbol{R}_i\boldsymbol{Z}_x)\boldsymbol{e}^{\boldsymbol{S}_r x}$

$\boldsymbol{G}_1 = -\boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{Q}_{11}\end{bmatrix}$, $\boldsymbol{G}_2 = -\boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{D}_{21}\end{bmatrix}$, $\boldsymbol{G}_3 = \boldsymbol{V}^{-1}\left(\boldsymbol{U}\boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\end{bmatrix} - \begin{bmatrix}\boldsymbol{B}_{12}\\ \boldsymbol{0}\end{bmatrix}\right)$, $\boldsymbol{G}_4 = \boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\end{bmatrix}$

$\boldsymbol{G}_5 = \boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\end{bmatrix}$, $\boldsymbol{G}_6 = \boldsymbol{V}^{-1}\left((\boldsymbol{T}\boldsymbol{V}^{-1} + (\boldsymbol{U}\boldsymbol{V}^{-1})^2)\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\boldsymbol{p}_1\end{bmatrix} - \boldsymbol{U}\boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{B}_{12}\boldsymbol{p}_1\\ \boldsymbol{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{p}_2\end{bmatrix}\right)$

$\boldsymbol{G}_7 = \boldsymbol{V}^{-1}\left(\boldsymbol{U}\boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\boldsymbol{p}_1\end{bmatrix} - \begin{bmatrix}\boldsymbol{B}_{12}\boldsymbol{p}_1\\ \boldsymbol{0}\end{bmatrix}\right)$, $\boldsymbol{G}_8 = \frac{1}{2}\boldsymbol{V}^{-1}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{B}_{21}^T\boldsymbol{B}_{11}\boldsymbol{p}_1\end{bmatrix}$

We express the limit conditions at the two extremities of the beam, in x = 0 and L:

$$\boldsymbol{\phi}_0 - \boldsymbol{G}_1\boldsymbol{\vartheta}_{q0} - \boldsymbol{G}_2\boldsymbol{\vartheta}_0 - \boldsymbol{G}_3\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right) - \boldsymbol{G}_5\boldsymbol{\Upsilon}_{l0} - \boldsymbol{G}_6 N = \boldsymbol{W}_0\boldsymbol{a} \quad (81)$$

$$\boldsymbol{\phi}_L - \boldsymbol{G}_1\boldsymbol{\vartheta}_{q0} - \boldsymbol{G}_2\boldsymbol{\vartheta}_0 - (\boldsymbol{G}_3 + \boldsymbol{G}_4 L)\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right) - \boldsymbol{G}_5\boldsymbol{\Upsilon}_{l0} - (\boldsymbol{G}_6 + \boldsymbol{G}_7 L + \boldsymbol{G}_8 L^2)N = \boldsymbol{W}_L\boldsymbol{a} \quad (82)$$

We assemble (81) and (82) to obtain the expression of the vector $\boldsymbol{a}$:

$$\boldsymbol{a}=\begin{bmatrix}\boldsymbol{W}_0\\ \boldsymbol{W}_L\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} & -\boldsymbol{G}_1 & -\boldsymbol{G}_2 & -\boldsymbol{G}_3 & -\boldsymbol{G}_5 & -\boldsymbol{G}_6\\ \boldsymbol{0} & \boldsymbol{I} & -\boldsymbol{G}_1 & -\boldsymbol{G}_2 & -(\boldsymbol{G}_3+\boldsymbol{G}_4L) & -\boldsymbol{G}_5 & -(\boldsymbol{G}_6+\boldsymbol{G}_7L+\boldsymbol{G}_8L^2)\end{bmatrix}\begin{Bmatrix}\boldsymbol{\phi}_0\\ \boldsymbol{\phi}_L\\ \boldsymbol{\vartheta}_{q0}\\ \boldsymbol{\vartheta}_0\\ \boldsymbol{\Gamma}_{q0}-\boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0}-\boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\\ \boldsymbol{\Upsilon}_{l0}\\ N\end{Bmatrix}$$

Thus, we can express the vector $\boldsymbol{a}$ in function of the boundary values:

$$\boldsymbol{a}=\boldsymbol{H}_0\boldsymbol{\phi}_0+\boldsymbol{H}_L\boldsymbol{\phi}_L+\boldsymbol{H}_1\boldsymbol{\vartheta}_{q0}+\boldsymbol{H}_2\boldsymbol{\vartheta}_0+\boldsymbol{H}_3\big(\boldsymbol{\Gamma}_{q0}-\boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0}-\boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\big)+\boldsymbol{H}_4\boldsymbol{\Upsilon}_{l0}+\boldsymbol{H}_5N \tag{83}$$

And by noting: $\boldsymbol{H}_{2\phi}\boldsymbol{\phi}_0=[\boldsymbol{0}\quad \boldsymbol{H}_2]\begin{Bmatrix}\boldsymbol{\varphi}_0\\ \boldsymbol{\vartheta}_0\end{Bmatrix}$, $\boldsymbol{G}_{2\phi}=[\boldsymbol{0}\quad \boldsymbol{G}_2]$, $\boldsymbol{H}_{3\phi}=[\boldsymbol{H}_3\boldsymbol{Q}_{12}^T\quad \boldsymbol{0}]$, $\boldsymbol{G}_{i\phi}=[\boldsymbol{G}_i\boldsymbol{Q}_{12}^T\quad \boldsymbol{0}]$ $i=3,4$

We can re-express our solution in the following form:

$$\begin{aligned}\boldsymbol{\phi}_x=\Big(\boldsymbol{G}_{2\phi}+\boldsymbol{G}_{3\phi}+\boldsymbol{G}_{4\phi}x+\boldsymbol{W}_x\big(\boldsymbol{H}_0+\boldsymbol{H}_{2\phi}+\boldsymbol{H}_{3\phi}\big)\Big)\boldsymbol{\phi}_0+\boldsymbol{W}_x\boldsymbol{H}_L\boldsymbol{\phi}_L+(\boldsymbol{G}_1+\boldsymbol{W}_x\boldsymbol{H}_1)\boldsymbol{\vartheta}_{q0}\\ +(\boldsymbol{G}_3+\boldsymbol{G}_4x+\boldsymbol{W}_x\boldsymbol{H}_3)\big(\boldsymbol{\Gamma}_{q0}-\boldsymbol{Q}_{11}^T\,\boldsymbol{\varphi}_{l0}\big)+(\boldsymbol{G}_5+\boldsymbol{W}_x\boldsymbol{H}_4)\boldsymbol{\Upsilon}_{l0}\\ +(\boldsymbol{G}_6+\boldsymbol{G}_7x+\boldsymbol{G}_8x^2+\boldsymbol{W}_x\boldsymbol{H}_5)N\end{aligned} \tag{84}$$

$$\boldsymbol{\phi}_x=\boldsymbol{E}_{\eta x}\boldsymbol{\eta}_{0L}+\boldsymbol{E}_{\Xi x}\Xi_{0L} \tag{85}$$

Where: $\boldsymbol{\eta}_x=\begin{Bmatrix}u\\ \boldsymbol{\vartheta}_q\\ \boldsymbol{\varphi}_l\\ \boldsymbol{\phi}\end{Bmatrix}$, $\boldsymbol{\eta}_{0L}=\begin{Bmatrix}\boldsymbol{\eta}_0\\ \boldsymbol{\eta}_L\end{Bmatrix}$, $\Xi_x=\begin{Bmatrix}N\\ \boldsymbol{\Gamma}_q\\ \boldsymbol{\Upsilon}_l\\ \boldsymbol{\Upsilon}\\ \boldsymbol{\Gamma}\end{Bmatrix}$, $\Xi_{0L}=\begin{Bmatrix}\Xi_0\\ \Xi_L\end{Bmatrix}$

$$\boldsymbol{E}_{\eta x}=\big[\boldsymbol{0}\quad \boldsymbol{G}_1+\boldsymbol{W}_x\boldsymbol{H}_1\quad -(\boldsymbol{G}_3+\boldsymbol{G}_4x+\boldsymbol{W}_x\boldsymbol{H}_3)\boldsymbol{Q}_{11}^T\quad \boldsymbol{G}_{23}+\boldsymbol{G}_{4\phi}x+\boldsymbol{W}_x\boldsymbol{H}_{023}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{W}_x\boldsymbol{H}_L\big] \tag{86}$$

$$\boldsymbol{E}_{\Xi x}=[(\boldsymbol{G}_6+\boldsymbol{G}_7x+\boldsymbol{G}_8x^2+\boldsymbol{W}_x\boldsymbol{H}_5)\quad \boldsymbol{G}_3+\boldsymbol{G}_4x+\boldsymbol{W}_x\boldsymbol{H}_3\quad \boldsymbol{G}_5+\boldsymbol{W}_x\boldsymbol{H}_4\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{0}\quad \boldsymbol{0}] \tag{87}$$

And : $\boldsymbol{H}_{023}=\boldsymbol{H}_0+\boldsymbol{H}_{2\phi}+\boldsymbol{H}_{3\phi}$, $\boldsymbol{G}_{23}=\boldsymbol{G}_{2\phi}+\boldsymbol{G}_{3\phi}$

We note:

$$\boldsymbol{\phi}_{-1x}=\int_0^x\boldsymbol{\phi}_t dt=\boldsymbol{E}_{-1,\eta x}\boldsymbol{\eta}_{0L}+\boldsymbol{E}_{-1,\Xi x}\Xi_{0L}\quad ,\quad \boldsymbol{E}_{-1,\eta x}=\int_0^x\boldsymbol{E}_{\eta t}dt\quad ,\quad \boldsymbol{E}_{-1,\Xi x}=\int_0^x\boldsymbol{E}_{\Xi t}dt \tag{88}$$

$$\boldsymbol{\phi}_{-2x} = \int_0^x \boldsymbol{\phi}_{-1t} dt = \boldsymbol{E}_{-2,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{-2,\Xi x}\boldsymbol{\Xi}_{0L} \quad , \quad \boldsymbol{E}_{-2,\eta x} = \int_0^x \boldsymbol{E}_{-1,\eta t} dt \quad , \quad \boldsymbol{E}_{-2,\Xi x} = \int_0^x \boldsymbol{E}_{-1,\Xi t} dt \tag{89}$$

$$\boldsymbol{\phi}_{1x} = \boldsymbol{\phi}'_x = \boldsymbol{E}_{1,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{1,\Xi x}\boldsymbol{\Xi}_{0L} \quad , \quad \boldsymbol{E}_{1,\eta x} = \frac{d\boldsymbol{E}_{\eta x}}{dx}, \quad \boldsymbol{E}_{1,\Xi x} = \frac{d\boldsymbol{E}_{\Xi x}}{dx} \tag{90}$$

For the calculation of the integrals and derivative of $\boldsymbol{\phi}_x$, see appendix 5.

By using the solution of the system, we obtain $\boldsymbol{\vartheta}$ that we can replace in the 3rd equation of (62) to obtain $\boldsymbol{\varphi}_l'$ and by integrating we obtain $\boldsymbol{\varphi}_l$ in function of the boundary conditions:

$$\boldsymbol{\varphi}_l' = \boldsymbol{B}_{21}\boldsymbol{\vartheta} + \boldsymbol{h}_x$$

$$\boldsymbol{\varphi}_l = \boldsymbol{\varphi}_{l0} + \boldsymbol{h}_{-1x} + \boldsymbol{B}_{21}\int_0^x \boldsymbol{\vartheta}\, dt$$

$$\boldsymbol{\varphi}_l = \boldsymbol{\varphi}_{l0} + \boldsymbol{h}_{-1x} + \boldsymbol{B}_{21}\, \boldsymbol{A}_\vartheta \left(\boldsymbol{E}_{-1,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{-1,\Xi x}\boldsymbol{\Xi}_{0L}\right)$$

$$\Longrightarrow \quad \boldsymbol{\varphi}_l = \boldsymbol{F}_{\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{F}_{\Xi x}\boldsymbol{\Xi}_{0L} \tag{91}$$

Where: $\boldsymbol{A}_\vartheta = [\boldsymbol{0} \quad \boldsymbol{I}]$, $\boldsymbol{h}_{-1x} = \int_0^x \boldsymbol{h}_t dt = \left(\boldsymbol{\Upsilon}_{l0} - \boldsymbol{D}_{21}\boldsymbol{\vartheta}_0 - \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_{q0}\right)x + \boldsymbol{B}_{11}\left(\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\, \boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right)\frac{x^2}{2} + \boldsymbol{p}_1 N\frac{x^3}{6}\right)$

With $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}_l$ obtained, we can replace then in the 2nd equation of (62) to obtain $\boldsymbol{\vartheta}_q'$ and by integrating we obtain $\boldsymbol{\vartheta}_q$ in function of the boundary conditions:

$$\boldsymbol{\vartheta}_q' = \boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\, \boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\, \boldsymbol{\varphi}_0 - \boldsymbol{B}_{11}^T\boldsymbol{\varphi}_l - \boldsymbol{B}_{12}^T\boldsymbol{\varphi}$$

$$\boldsymbol{\vartheta}_q' = \boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\, \boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\, \boldsymbol{\varphi}_0 - \boldsymbol{B}_{11}^T(\boldsymbol{\varphi}_{l0} + \boldsymbol{h}_{-1x} + \boldsymbol{B}_{21}\boldsymbol{A}_\vartheta\, \boldsymbol{\phi}_{-1x}) - \boldsymbol{B}_{12}^T\boldsymbol{A}_\varphi\, \boldsymbol{\phi}_x$$

$$\boldsymbol{\vartheta}_q = \boldsymbol{\vartheta}_{q0} + \left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\, \boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\, \boldsymbol{\varphi}_0\right)x - \boldsymbol{B}_{11}^T\left(\boldsymbol{\varphi}_{l0}x + \boldsymbol{h}_{-2x} + \boldsymbol{B}_{21}^T\boldsymbol{A}_\vartheta\left(\boldsymbol{E}_{-2,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{-2,\Xi x}\boldsymbol{\Xi}_{0L}\right)\right)$$
$$- \boldsymbol{B}_{12}\boldsymbol{A}_\varphi\left(\boldsymbol{E}_{-1,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{-1,\Xi x}\boldsymbol{\Xi}_{0L}\right)$$

$$\Longrightarrow \quad \boldsymbol{\vartheta}_q = \boldsymbol{H}_{\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{H}_{\Xi x}\boldsymbol{\Xi}_{0L} \tag{92}$$

Where: $\boldsymbol{A}_\varphi = [\boldsymbol{I} \quad \boldsymbol{0}]$, $\boldsymbol{h}_{-2x} = \int_0^x \boldsymbol{h}_{-1t} dt = \left(\boldsymbol{\Upsilon}_{l0} - \boldsymbol{D}_{21}\boldsymbol{\vartheta}_0 - \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_{q0}\right)\frac{x^2}{2} + \boldsymbol{B}_{11}\left(\left(\boldsymbol{\Gamma}_{q0} - \boldsymbol{Q}_{11}^T\, \boldsymbol{\varphi}_{l0} - \boldsymbol{Q}_{12}^T\boldsymbol{\varphi}_0\right)\frac{x^3}{6} + \boldsymbol{p}_1 N\frac{x^4}{24}\right)$

The resolution is completed by performing a simple variable change to obtain the original vectors $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. To derive the rigidity matrix, we will as in Ferradi et al[1], assemble all the equilibrium equations that we will express at the two extremities of the beam, in x=0 and x=L.

From (46) and (58) we have:

$$\begin{cases} \boldsymbol{\Upsilon} = \boldsymbol{\varphi}' + \boldsymbol{Q}_{12}\boldsymbol{\vartheta}_q + \boldsymbol{Q}_{22}\boldsymbol{\vartheta} \\ \boldsymbol{\Gamma} = \boldsymbol{\vartheta}' + \boldsymbol{D}_{21}^T\boldsymbol{\varphi}_l + \boldsymbol{D}_{22}^T\boldsymbol{\varphi} \end{cases} \Rightarrow \begin{Bmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\Gamma} \end{Bmatrix} = \boldsymbol{E}_{1,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{1,\Xi x}\Xi_{0L} + \begin{bmatrix} \mathbf{0} & \boldsymbol{Q}_{12} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{22} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{D}_{21}^T & \boldsymbol{D}_{22}^T & \mathbf{0} \end{bmatrix} \boldsymbol{\eta}_x \tag{93}$$

This system is expressed at the two extremities of the beam, at x=0 and x=L, so we obtain 2k equations:

$$\begin{Bmatrix} \boldsymbol{\Upsilon}_0 \\ \boldsymbol{\Gamma}_0 \\ \boldsymbol{\Upsilon}_L \\ \boldsymbol{\Gamma}_L \end{Bmatrix} - \begin{bmatrix} \boldsymbol{E}_{1,\Xi 0} \\ \boldsymbol{E}_{1,\Xi L} \end{bmatrix} \Xi_{0L} = \left(\begin{bmatrix} \boldsymbol{E}_{1,\eta 0} \\ \boldsymbol{E}_{1,\eta L} \end{bmatrix} + \begin{bmatrix} \begin{matrix} \mathbf{0} & \boldsymbol{Q}_{12} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{22} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{D}_{21}^T & \boldsymbol{D}_{22}^T & \mathbf{0} \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \mathbf{0} & \boldsymbol{Q}_{12} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{22} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{D}_{21}^T & \boldsymbol{D}_{22}^T & \mathbf{0} \end{matrix} \end{bmatrix} \right) \boldsymbol{\eta}_{0L} \tag{94}$$

We also have from (46) and (58) the expression of the generalized efforts vectors $\boldsymbol{\Gamma}_q$ and $\boldsymbol{\Upsilon}_l$, that we will express in function of the boundary values:

$$\begin{cases} \boldsymbol{\Gamma}_q = \boldsymbol{\vartheta}_q{}' + \boldsymbol{D}_{11}^T\boldsymbol{\varphi}_l + \boldsymbol{D}_{12}^T\boldsymbol{\varphi} \\ \boldsymbol{\Upsilon}_l = \boldsymbol{\varphi}_l{}' + \boldsymbol{Q}_{11}\boldsymbol{\vartheta}_q + \boldsymbol{Q}_{21}\boldsymbol{\vartheta} \end{cases} \Rightarrow \begin{Bmatrix} \boldsymbol{\Gamma}_q \\ \boldsymbol{\Upsilon}_l \end{Bmatrix} = \begin{bmatrix} \boldsymbol{H}_{1,\eta x} \\ \boldsymbol{F}_{1,\eta x} \end{bmatrix} \boldsymbol{\eta}_{0L} + \begin{bmatrix} \boldsymbol{H}_{1,\Xi x} \\ \boldsymbol{F}_{1,\Xi x} \end{bmatrix} \Xi_{0L} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{D}_{11}^T & \boldsymbol{D}_{12}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_{11} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{21} \end{bmatrix} \boldsymbol{\eta}_x \tag{95}$$

Where: $\boldsymbol{F}_{1,\eta x} = \frac{d\boldsymbol{F}_{\eta x}}{dx}$, $\boldsymbol{F}_{1,\Xi x} = \frac{d\boldsymbol{F}_{\Xi x}}{dx}$, $\boldsymbol{H}_{1,\eta x} = \frac{d\boldsymbol{H}_{\eta x}}{dx}$, $\boldsymbol{H}_{1,\Xi x} = \frac{d\boldsymbol{H}_{\Xi x}}{dx}$

We express (96) only at x=L, to obtain q+l equations:

$$\begin{Bmatrix} \boldsymbol{\Gamma}_{qL} \\ \boldsymbol{\Upsilon}_{lL} \end{Bmatrix} - \begin{bmatrix} \boldsymbol{H}_{1,\Xi L} \\ \boldsymbol{F}_{1,\Xi L} \end{bmatrix} \Xi_{0L} = \begin{bmatrix} \boldsymbol{H}_{1,\eta L} \\ \boldsymbol{F}_{1,\eta L} \end{bmatrix} \boldsymbol{\eta}_{0L} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{D}_{11}^T & \boldsymbol{D}_{12}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q}_{11} & \mathbf{0} & \mathbf{0} & \boldsymbol{Q}_{21} \end{bmatrix} \boldsymbol{\eta}_L \tag{96}$$

By expressing $\boldsymbol{\vartheta}_q$ and $\boldsymbol{\varphi}_l$ at x=L, we obtain the remaining q+l equations:

$$\begin{bmatrix} \boldsymbol{H}_{\Xi L} \\ \boldsymbol{F}_{\Xi L} \end{bmatrix} \Xi_{0L} = \left(\begin{bmatrix} \boldsymbol{H}_{\eta L} \\ \boldsymbol{F}_{\eta L} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \boldsymbol{\eta}_{0L} \tag{97}$$

And finally from the integration of the 1st equation in (62) and the equilibrium equation $\frac{dN}{dx} = 0$, we add the two following equations (98) and (99):

$$u_L - u_0 = \frac{N_0 L}{(2\mu+\lambda)S} - \boldsymbol{p}_1 \cdot \int_0^L \boldsymbol{\vartheta}_q dx - \boldsymbol{p}_2 \cdot \int_0^L \boldsymbol{\vartheta}\, dx$$

$$u_L - u_0 = \frac{N_0 L}{(2\mu+\lambda)S} - \boldsymbol{p}_1 \cdot \left(\boldsymbol{H}_{-1,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{H}_{-1,\Xi x}\Xi_{0L}\right) - \boldsymbol{p}_2 \cdot \boldsymbol{A}_{\vartheta}\left(\boldsymbol{E}_{-1,\eta x}\boldsymbol{\eta}_{0L} + \boldsymbol{E}_{-1,\Xi x}\Xi_{0L}\right)$$

$$\Longrightarrow\ u_L - u_0 + \left(\boldsymbol{H}^{\boldsymbol{T}}_{-1,\eta x}\,\boldsymbol{p}_1 + \boldsymbol{E}^{\boldsymbol{T}}_{-1,\eta x}\,\boldsymbol{A}^{\boldsymbol{T}}_{\vartheta}\boldsymbol{p}_2\right)\cdot \boldsymbol{\eta}_{0L} = \frac{N_0 L}{(2\mu+\lambda)S} - \left(\boldsymbol{H}^{\boldsymbol{T}}_{-1,\Xi x}\,\boldsymbol{p}_1 + \boldsymbol{E}^{\boldsymbol{T}}_{-1,\Xi x}\,\boldsymbol{A}^{\boldsymbol{T}}_{\vartheta}\boldsymbol{p}_2\right)\cdot \Xi_{0L} \tag{98}$$

$$N_L - N_0 = 0 \tag{99}$$

Where: $\boldsymbol{F}_{-1,\eta x} = \int_0^x \boldsymbol{F}_{\eta t} dt$, $\boldsymbol{F}_{-1,\Xi x} = \int_0^x \boldsymbol{F}_{\Xi t} dt$, $\boldsymbol{H}_{-1,\eta x} = \int_0^x \boldsymbol{H}_{\eta t} dt$, $\boldsymbol{H}_{-1,\Xi x} = \int_0^x \boldsymbol{H}_{\Xi t} dt$

Thus, by assembling all these equations we obtain a system of 2(m+n+1) equations, written in the following form:

$$\boldsymbol{K}_{\Xi}\,\Xi_{0L} = \boldsymbol{K}_{\eta}\,\boldsymbol{\eta}_{0L} \quad \Longrightarrow \quad \Xi_{0L} = \underbrace{\boldsymbol{K}_{\Xi}^{-1}\,\boldsymbol{K}_{\eta}}_{\boldsymbol{K}_r}\,\boldsymbol{\eta}_{0L} \tag{100}$$

$\boldsymbol{K}_r$ is then the rigidity matrix of the beam element, derived from the exact solution of the equilibrium equation. This matrix was implemented in Pythagore™ software.

For quasi-incompressible materials ($\upsilon \mapsto 0.5 \implies \lambda \mapsto +\infty$), a special attention must be given to the transversal modes we are using in our kinematics to avoid incompressible locking. In the expression of the stress tensor in eq.23, we have the apparition of the stress $\sigma_r = \lambda tr(\boldsymbol{\varepsilon}) = \lambda\left(\frac{du}{dx} + \zeta^i div\,\boldsymbol{\psi}^i + \Omega_j \frac{d\xi_j}{dx}\right)$, thus when $\lambda \mapsto +\infty$ we need to be able represent $tr(\boldsymbol{\varepsilon}) = 0$, and this can be assured by associating to every warping mode Ω_j, a transversal deformation mode $\boldsymbol{\psi}^{j*}$, constructed in a way to compensate the warping mode, and thus to satisfy $tr(\boldsymbol{\varepsilon}) = 0$ when it's needed. These newly determined transversal modes will be called 'incompressible deformation modes' and will verify the relation $div\,\boldsymbol{\psi}^{j*} = \Omega_j$. We have also noticed that even for $0 < \upsilon < 0.5$ (for example υ = 0.3 for steel) the error in the results cannot be negligible if we do not consider the 'incompressible deformation modes', thus we have chosen in all our examples to take υ = 0 to avoid any discrepancy, noting that this subject will be treated in detail in upcoming work.

5. THE GENERALIZED EFFORT VECTOR :

We have the displacement vector of an arbitrary point P of the section:

$$\boldsymbol{d}_p = \begin{Bmatrix} u_p \\ v_p \\ w_p \end{Bmatrix} = \begin{Bmatrix} u + \sum_{j=1}^{m} \Omega_j \xi_j \\ \sum_{i=1}^{n} \psi_y^i \zeta^i \\ \sum_{i=1}^{n} \psi_z^i \zeta^i \end{Bmatrix} \quad \Rightarrow \quad \boldsymbol{d}_p = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 & \Omega_1 & \dots & \Omega_m \\ 0 & \psi_y^1 & \dots & \psi_y^n & 0 & \dots & 0 \\ 0 & \psi_z^1 & \dots & \psi_z^n & 0 & \dots & 0 \end{bmatrix}}_{\boldsymbol{F}} \underbrace{\begin{Bmatrix} u \\ \zeta^1 \\ \vdots \\ \zeta^n \\ \xi_1 \\ \vdots \\ \xi_m \end{Bmatrix}}_{\boldsymbol{d}} \tag{101}$$

Where **d** is the vector representing the generalized coordinates.

If we apply at the point P a force represented by its vector $\boldsymbol{f}_p$, we will have the following equivalent generalized stress resultants :

$$\boldsymbol{f}_p = \begin{Bmatrix} N \\ T_y \\ T_z \end{Bmatrix} \quad \Rightarrow \quad \boldsymbol{f} = \boldsymbol{F}^T \boldsymbol{f}_p \quad \Rightarrow \quad \boldsymbol{f} = \begin{Bmatrix} N \\ \psi_y^1 T_y + \psi_z^1 T_z \\ \vdots \\ \psi_y^n T_y + \psi_z^n T_z \\ \Omega_1 N \\ \vdots \\ \Omega_m N \end{Bmatrix} \tag{102}$$

6. NUMERICAL EXAMPLES :

In all the examples presented in this section, we will perform three types of comparisons, one concerns the comparison of the displacement at the section where the load is applied and where the effect of the higher transversal modes will be important, the 2nd concerns the normal stress at x=0.05m the vicinity of the fixed end, where the effect of restrained warping is the most important and the effect of the higher warping modes are non-negligible and the 3rd concerns the shear stress in the upper slab at x=0.95m.

For all the figures representing the numerical results, the table 1 and 2 give the transversal modes used for each beam model in the figure and the number of corresponding warping modes used for each transversal mode and gives also the type of cross section used in the beam models, 1D if the cross section is discretized with 1D (or beam) element and 2D if it's discretized with triangular element.

Box girder:

We consider in the following examples a 10m length cantilever beam, with E=40 GPa and υ=0, loaded at its free end. The comparisons will be performed between a shell model of the cantilever beam using the well-known MITC- 4 shell element described in [8], and a model with just *one element* of the new beam finite element, using different numbers of transversal modes. All comparisons have been performed with Pythagore™ Software.

The boundary conditions of the considered example are:

- No displacement at the beam's bearing: $u = 0$, $\xi_j = 0$, $\zeta^i = 0$ for all warping and transversal modes.
- At the free end: $\frac{d\xi_j}{dx} = 0, \frac{d\zeta^i}{dx} = 0.$

For the beam cross section we choose a box girder represented in figure 3. Some numerical values for the matrices in p.12 are given in appendix A7.

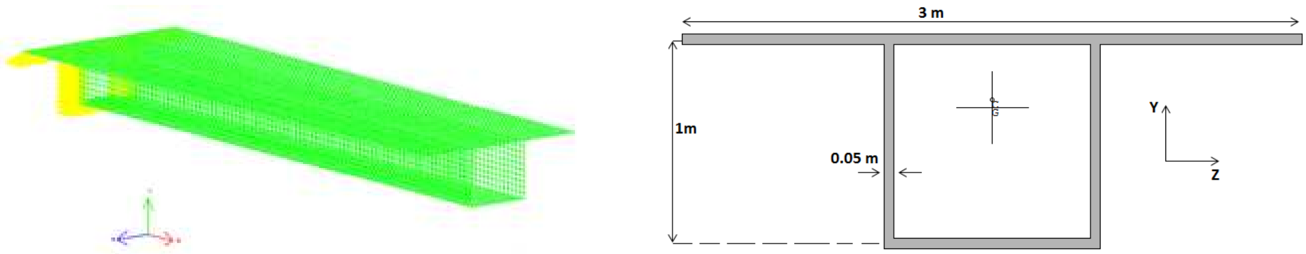


Figure 3 : A view of the shell model of the beam and its cross section.

The beam cross section will be discretized in two manners, the 1st one with 7024 2D triangular elements, and the 2nd one as a thin walled section with 6 1D elements. For the representation of the warping and transversal modes see appendix A6.

For the shell model of the beam, to obtain accurate results we use a refined meshing, where the dimension of the small square element is 0.1m, the model will then comport 6000 shell element, 6060 nodes and 36360 degree of freedom. To measure the effect of the meshing refinement and its necessity for the shell model of the beam, we will compare the different results obtained from shell models with different meshing.

We consider as a 1st load case a centred force T_y=-1MN at the upper slab of the box cross section and at the free end of the cantilever beam, see figure 4.

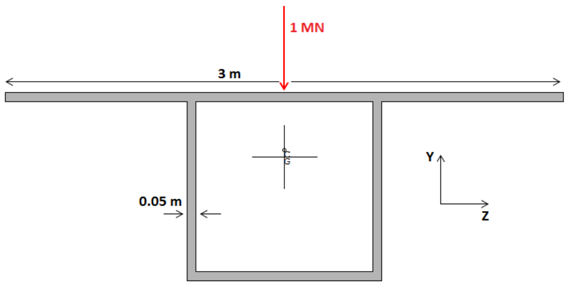


Figure 4: beam cross section with the centred applied load.

	Beam model (type of cross section)	Transversal modes	Corresponding warping modes
Figure 5	A (1D)	1	5
		2-3	1
Figure 6	A(2D)	1	4
		2-3	1
	B(2D)	1	2
		2-3	1
Figure 7	A(2D)	1	2
		2-10	1
	B(2D)	1	2
		2-12	1
	C(2D)	1	2
		2-20	1
	D(2D)	1	2
		2-3	1
Figure 8	A(2D)	1	2
		2-3	1
	B(2D)	1	2
		2-20	1

	Beam model (type of cross section)	Transversal modes	Corresponding warping modes
Figure 12	A(2D)	1	2
		2-3	1
	B(2D)	1	2
		2-6	1
	C(2D)	1	2
		2-10	1
Figure 14	A(2D) ; B(1D)	1	5
		2	1
		3	4
Figure 15	A(2D)	1	5
		2	1
		3	4
		4-6	1
	B(2D)	1	2
		2-3	1

Table 1: the transversal modes with their corresponding warping modes for the beam models used in the listed figures.

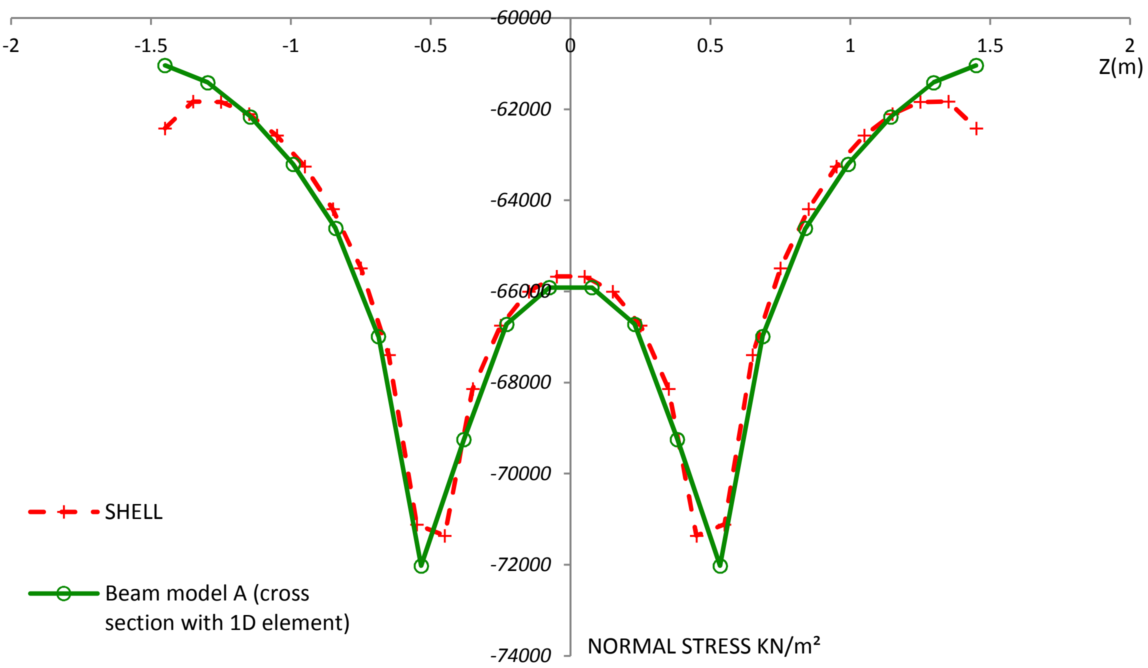


Figure 5 : Comparison of the normal stresses between the shell and the beam model, at x= 0.05m and at mid-depth of the upper slab (Ty = -1 MN)

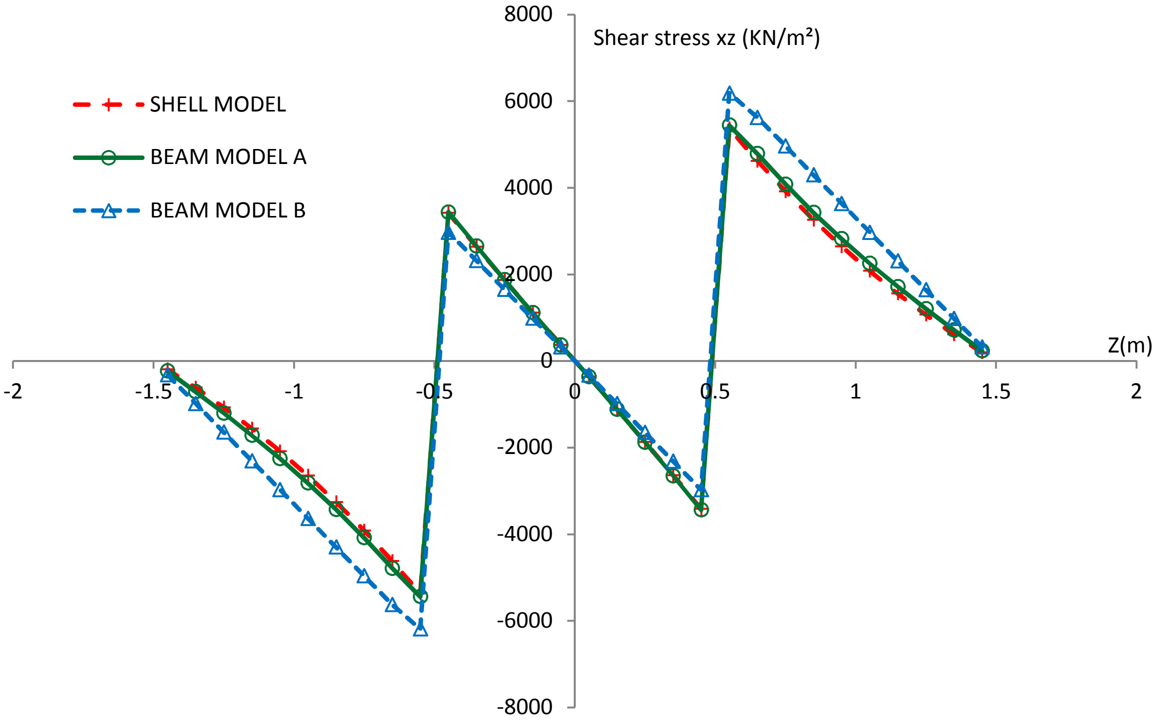


Figure 6 : Comparison of the shear stress xz between the shell and the beam models, at x= 0.95m and at mid-depth of the upper slab (Ty = -1 MN)

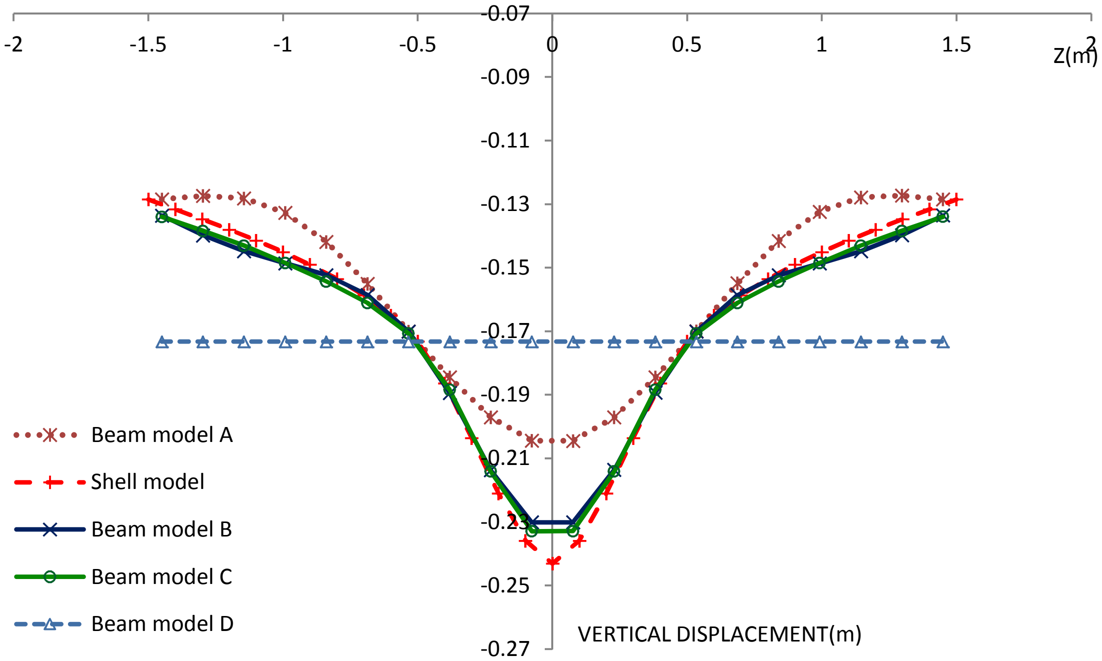


Figure 7 : Comparison of the displacement between the shell and the beam models, at x= 10m and at mid-depth of the upper slab (Ty = -1 MN)

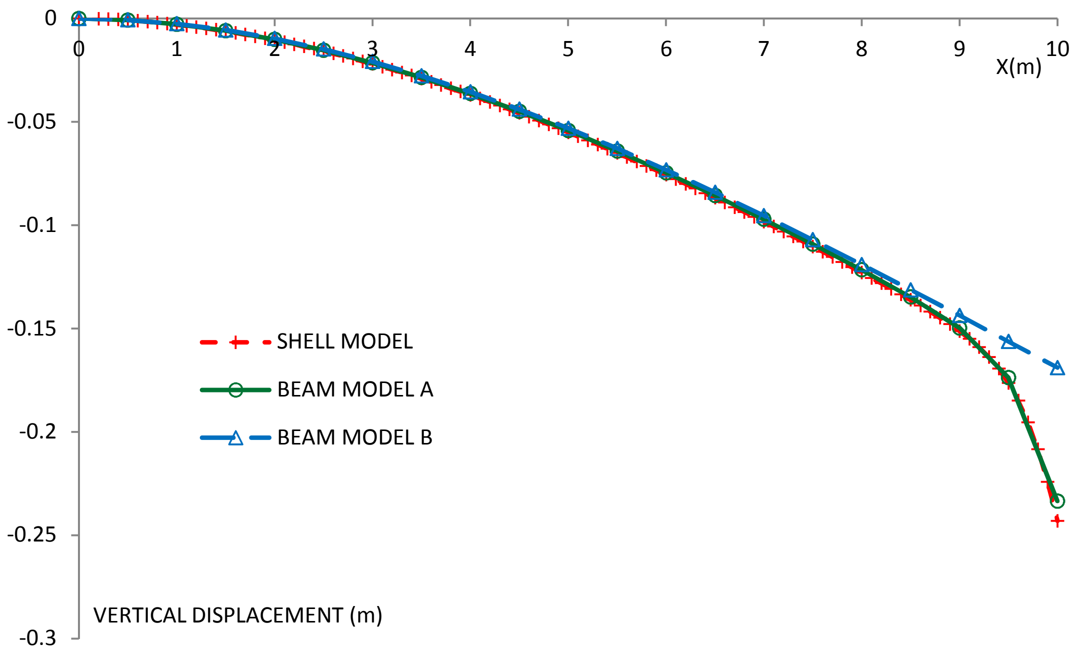


Figure 8 : Comparison of the vertical displacement along the beam length between the shell and the beam models.

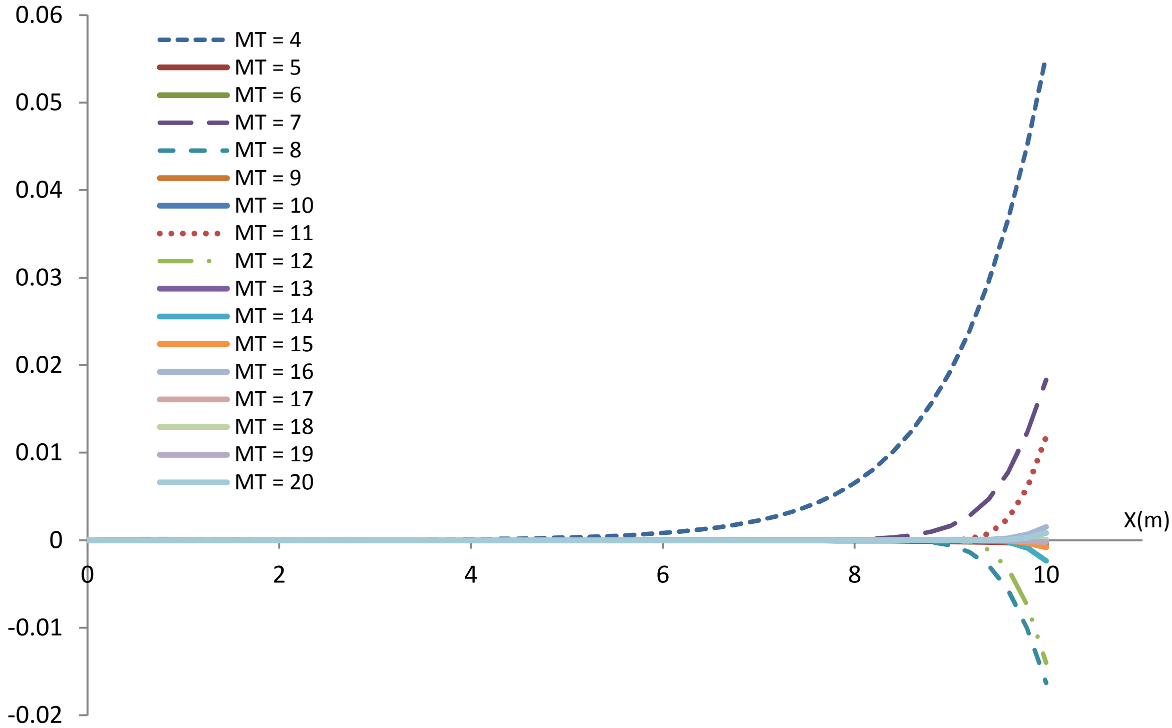


Figure 9 : representation of the generalized coordinates associated with the higher transversal modes(4-20) along the beam's length, for the beam model C in Fig.7.

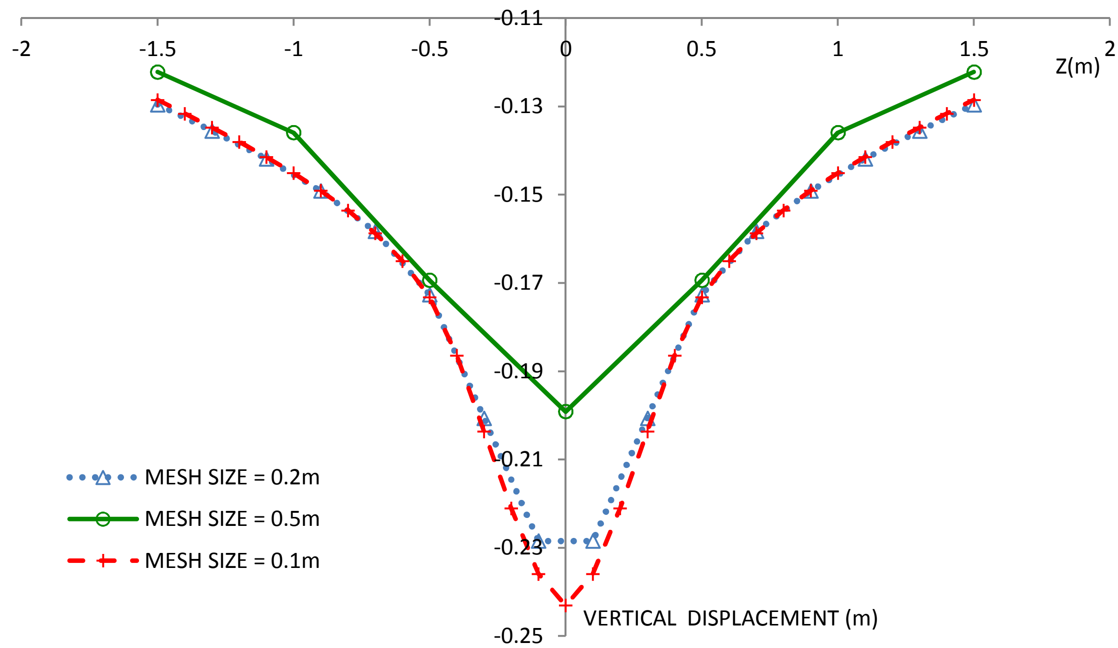


Figure 10 : Comparison of the results obtained from the shell model with different mesh size.

The Fig.5 illustrate the fact that in this example the shear lag effect near the fixed end is correctly predicted by using five warping modes for the vertical displacement mode. Fig.6 shows that we have a precise representation of shear stress distribution with only three transversal modes (rigid body motion) and four warping modes associated to the vertical displacement mode.

In Fig. 7, we can see that the beam model B (12 transversal modes) gives satisfactory results that still can be refined by using more transversal modes, and this is performed with the beam model C (20 transversal modes). We can see also from Fig.7 that at the connections between the upper slab and the two webs (z=-0.5m and z=0.5m), the vertical displacement is exactly the one corresponding to a rigid body motion, and this is valid for the shell and all the beam models. The figures 8 and 9 shows that the effect of the higher order transversal modes become more and more important when we approach the loading zone, Fig.8 shows also that we obtain a precise description of the vertical displacement distribution along the beam, especially near the loading zone.

From Fig.10, we deduce that the refined shell model of the beam, with a mesh size equal to 0.1m, is necessary to obtain accurate prediction of the vertical displacement at the loading cross section. This model has 6000 shell elements, 6060 nodes, with a total of 36360 d.o.f., and compared to the beam model B with *one beam element* and twelve transversal modes, with a total of 50 d.o.f., it shows the advantage of using such enriched beam element, even if it necessitate some preliminary cross section treatment to obtain the transversal and warping modes characteristics, because this step will be done only once and we can after perform as many calculation as we want with different loadings and configurations with the same reduced number of degree of freedom. We can see also that solving the equilibrium equations exactly, allows us to obtain accurate results without using a refined meshing of our beam element, and this is showed in the numerical examples where only one beam element is needed.

As a 2nd load case, we apply the same vertical force T_y=-1MN at the upper slab but eccentric from the centre of gravity and torsion (z=0.5m), see figure 11.

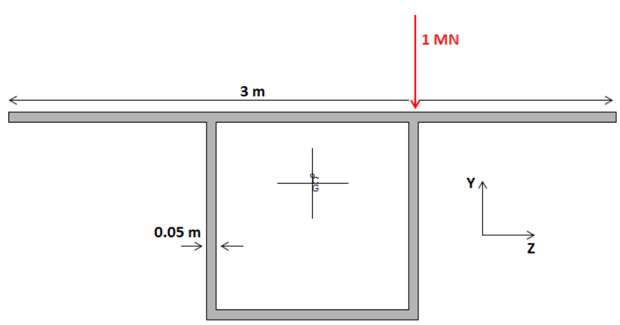


Figure 11: beam cross section with the excentred applied load.

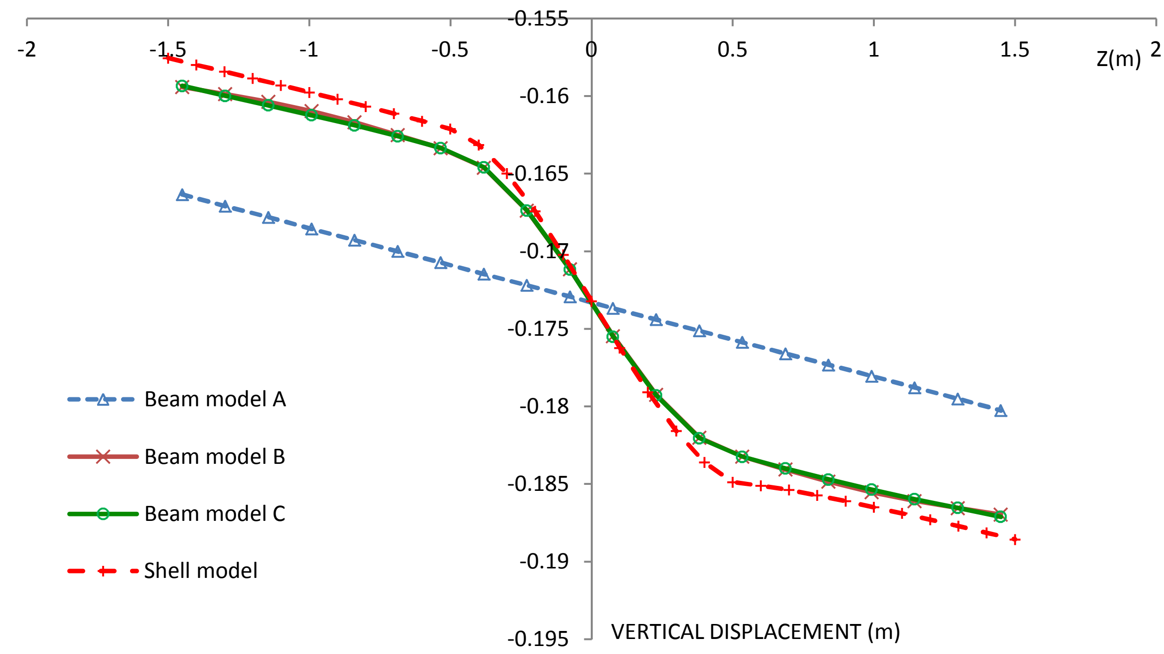


Figure 12 : Comparison of the displacement between the shell and the beam models, at x= 10m and at mid-depth of the upper slab (Ty = -1 MN)

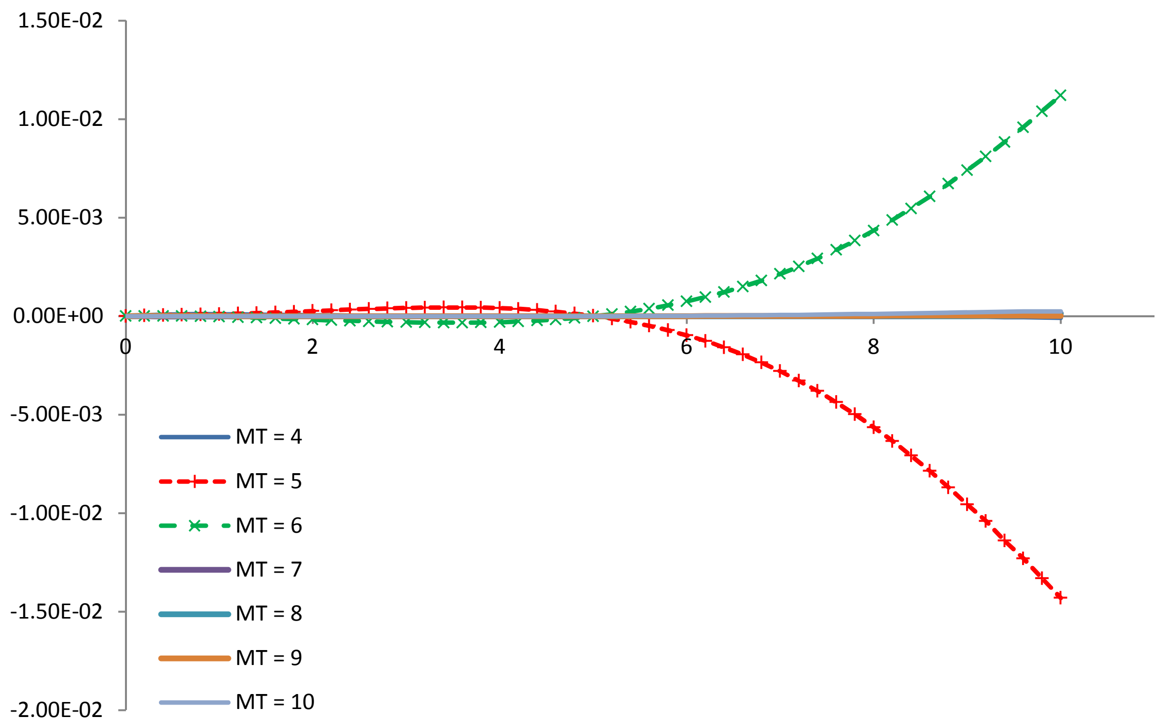


Figure 13 : representation of the generalized coordinates associated to the higher transversal modes(4-10) along the beam's length, for the beam model B in Fig.12.

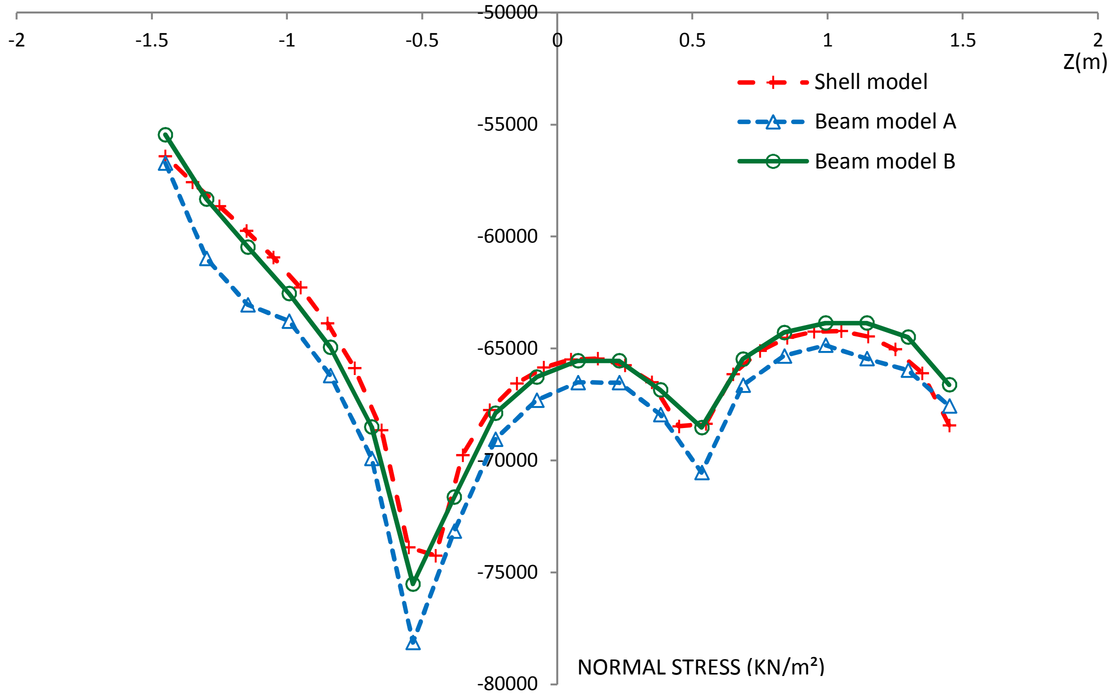


Figure 14 : Comparison of the normal stresses between the shell and the beam model, at x= 0.05m and at mid-depth of the upper slab (Ty = -1 MN)

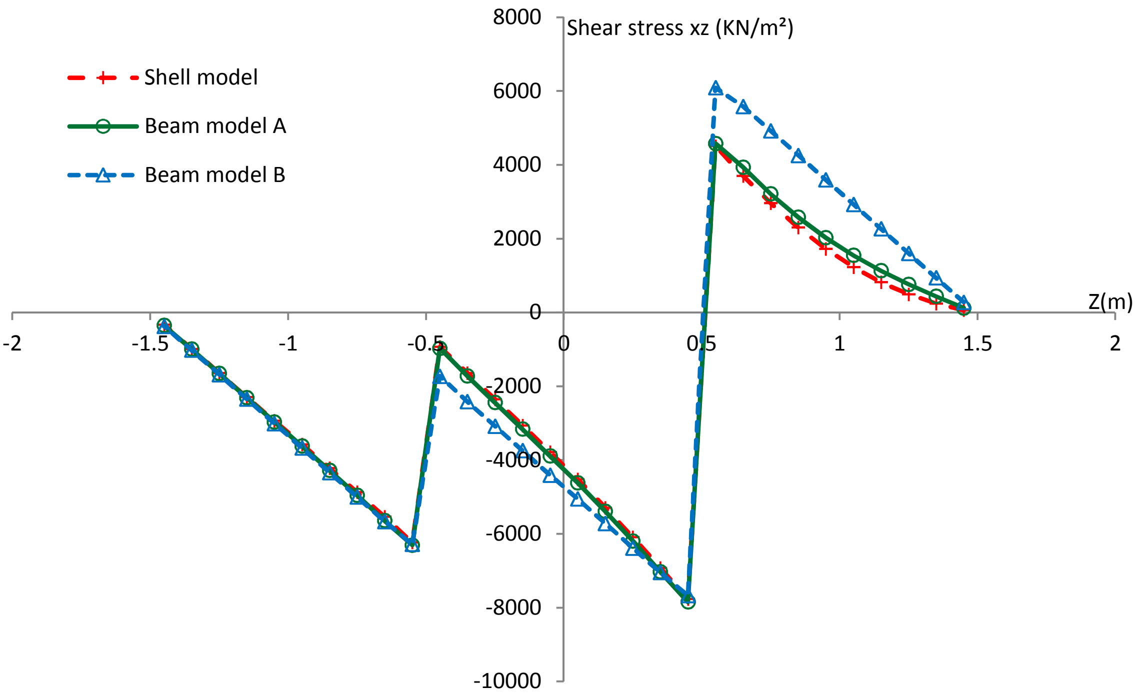


Figure 15 : Comparison of the shear stress xz between the shell and the beam models, at x= 0.95m and at mid-depth of the upper slab (Ty = -1 MN)

From Fig.12, we obtain satisfactory result with only 6 transversal modes, and always with just *one beam element*, which corresponds to a model with a total of 28 d.o.f. The Fig.13 shows that only the 5th and the 6th transversal modes give a substantial contribution to represent the beam behaviour; this is in accordance with the results of Fig.12, where there is no difference in the displacement distribution obtained with a beam model enriched with 6 or 10 transversal modes.

The figures 14 and 15 compare the stress distribution between shell and beam models, the results clearly show the effectiveness of the enriched beam model.

Double T cross section:

In this example we consider a 12m beam clamped at its both end and loaded with an eccentric (z=2m) vertical force of 10MN in its middle(x=6m), see Fig.16, the material characteristics will be the same as the previous example. We will use *two beam finite elements* to discretize the whole beam. To test the performance of our beam element, we will perform a comparison with a shell (MITC-4 element) and a brick (SOLID186 in Ansys™) model of the beam (Fig.17).

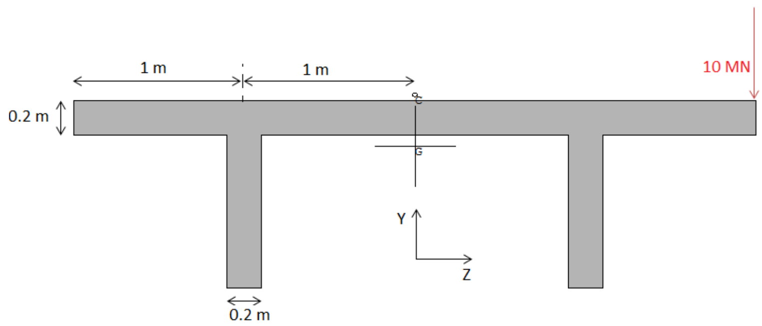


Figure 16: A view of the beam cross section.

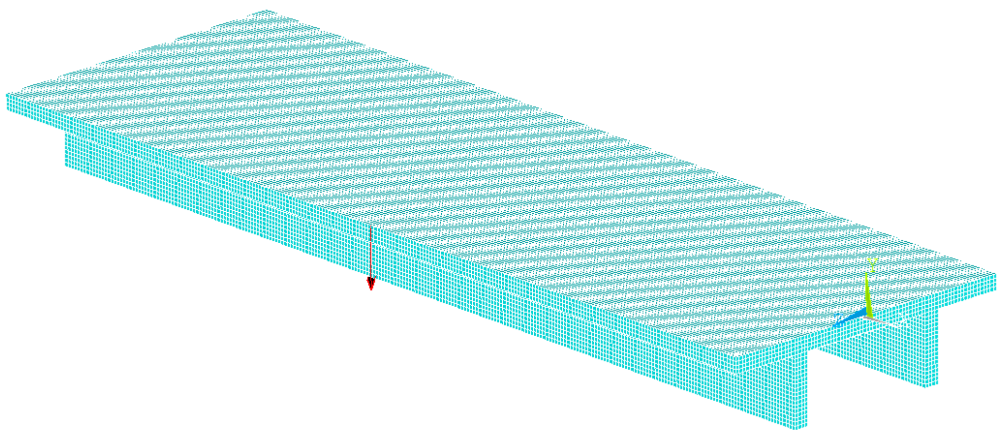


Figure 17: A view of the brick model of the beam.

	Beam model (type of cross section)	Transversal modes	Corresponding warping modes
Figure 18	A (1D)	1	2
		2-3	1
	B(1D)	1	2
		2-6	1
	C(1D)	1	2
		2-10	1
	D(2D)	1	2
		2-10	1
Figure 19	A(1D)	1	4
		2	1
		3	4
	B(2D)	1	4
		2	1
		3	4
Figure 20	A(1D)	1	4
		2	1
		3	4
	B(2D)	1	4
		2	1
		3	4

Table 2: the transversal modes with their corresponding warping modes for the beam models used in the listed figures.

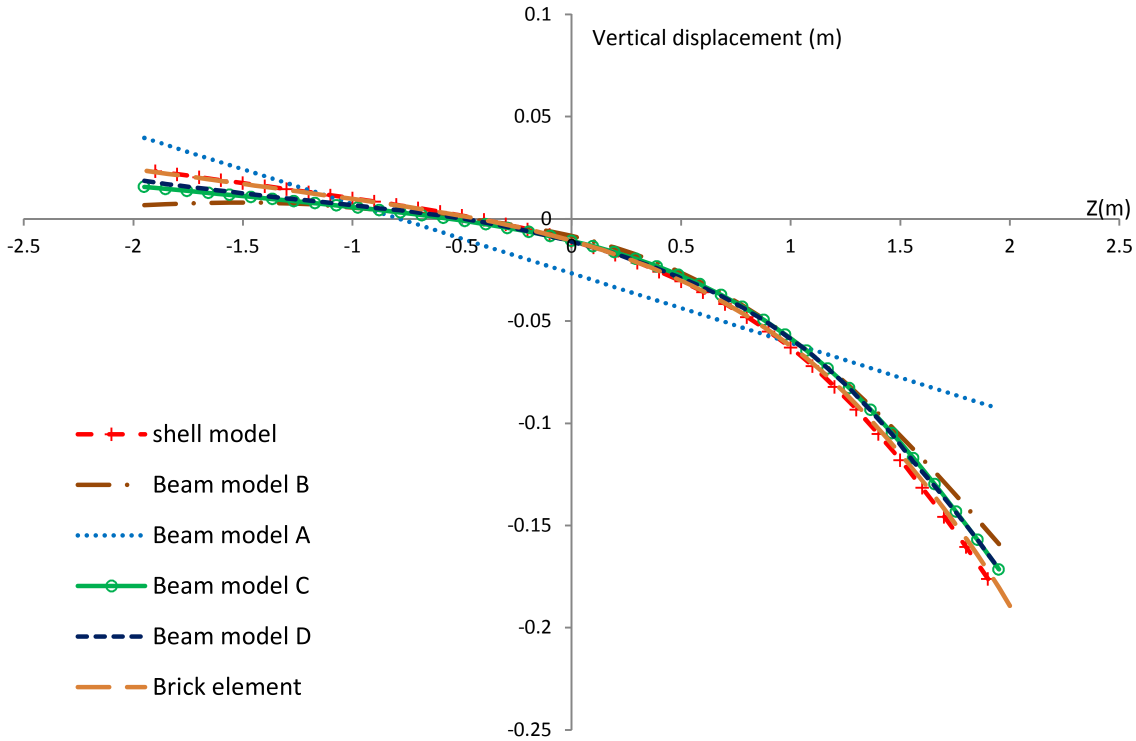


Figure 18: Comparison of the displacement between the brick, shell and the beam models, at x= 6m and at mid-depth of the upper slab (Ty = -10 MN)

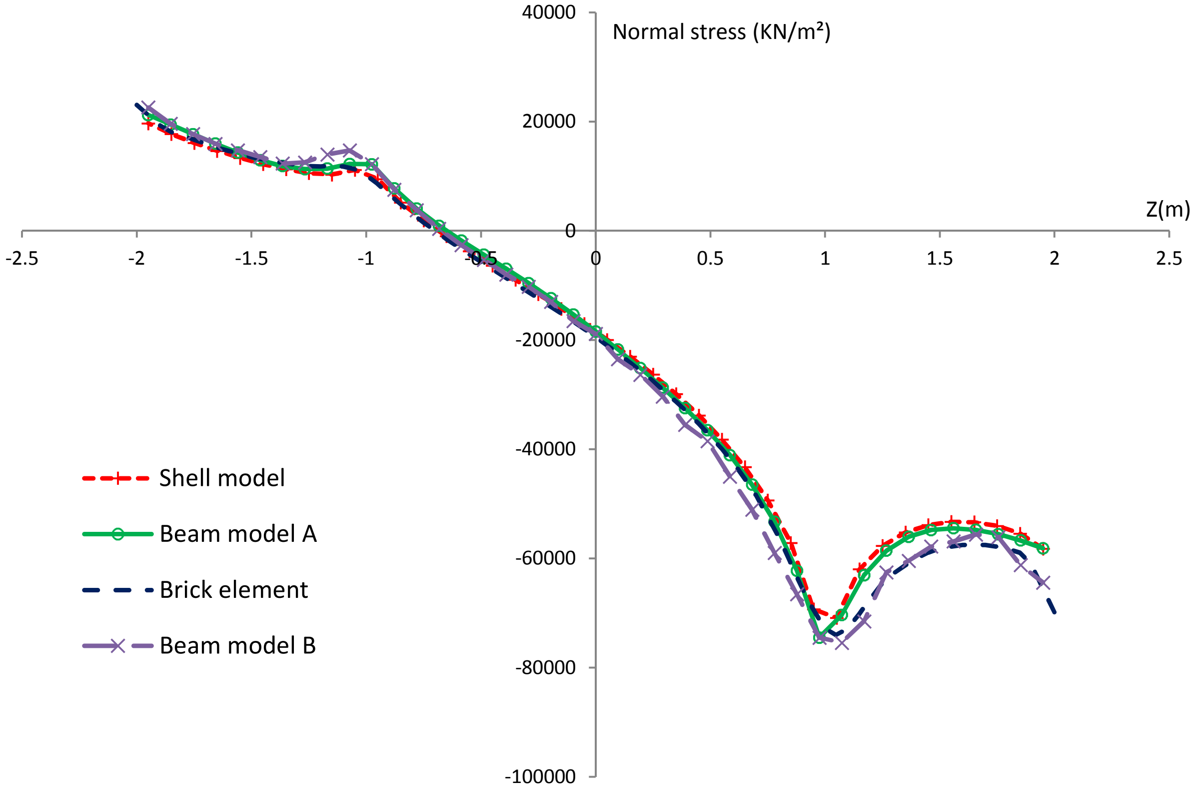


Figure 19 : Comparison of the normal stresses between the brick, shell and beam model, at x= 0.05m and at mid-depth of the upper slab (Ty = -10 MN)

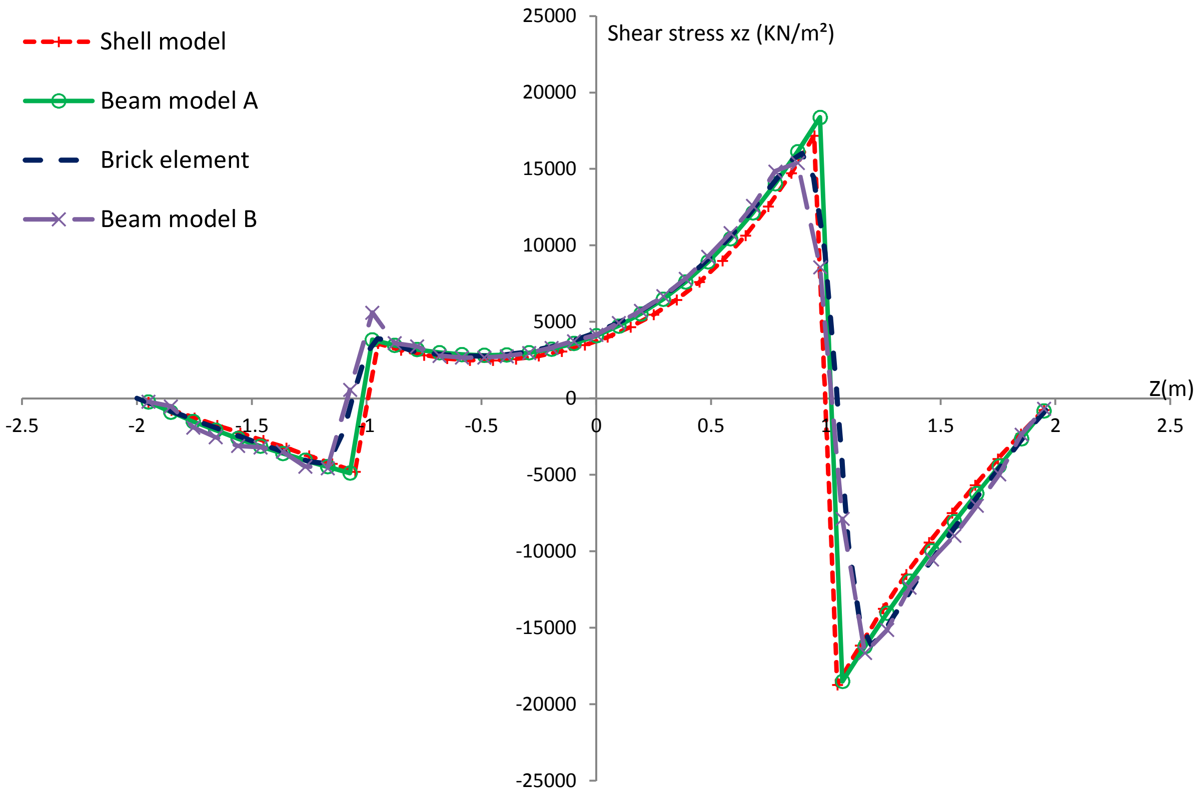


Figure 20 : Comparison of the shear stress xz between the brick shell and beam models, at x= 0.95m and at mid-depth of the upper slab (Ty = -10 MN)

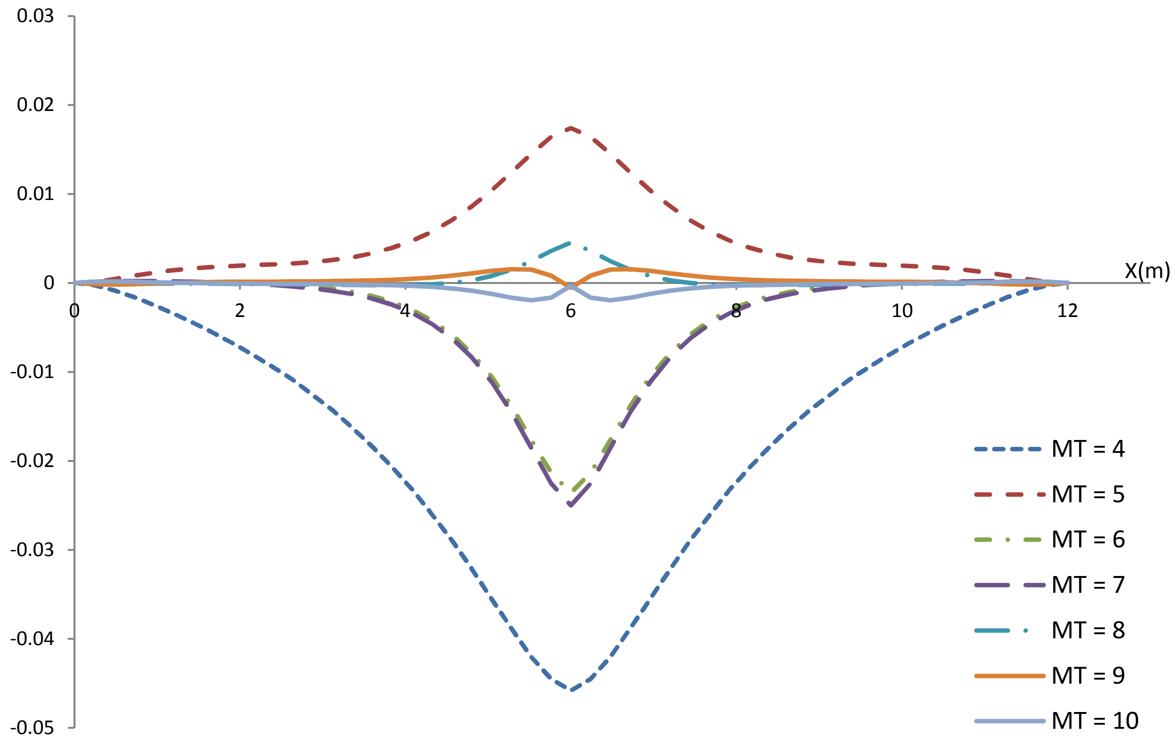


Figure 21 : representation of the generalized coordinates associated to the higher transversal modes(4-10) along the beam's length, for the beam model C in Fig.18.

The figures 18, 19 and 20 illustrate the comparison between a brick & shell models of the beam with our beam finite element models, showing the efficiency and the accuracy of our formulation. We note from Fig. 19 and 20 that the beam models B, with a cross section meshed with 2D triangular element is more close to the brick model. The beam models A, with a cross section meshed with 1D elements can be more closely compared to the shell model.

7. CONCLUSION:

In this work, we have presented a new beam finite element based on a new enhanced kinematics, enriched with transversal and warping eigenmodes, capable of representing the deformation and the displacement field of the beam in a very accurate way. A complete description of the method to derive these modes for an arbitrary section form is given. Theoretically we can determine as many warping modes as we want to enrich our kinematics, but we have observed that the number of modes that we can derive accurately is limited, due to the numerical errors accumulated in every new iteration of the iterative equilibrium process.

The additional transversal and warping modes give rise to new equilibrium equations associated with the newly introduced d.o.f. corresponding to each mode, forming a system of differential equations that can be assimilated to a one obtained from a gyroscopic system in an unstable state. An exact solution of these equations is performed, leading to the formulation of the rigidity matrix of a mesh free element. However we must note a limitation to this formulation: we need to calculate the exponential of a scalar (corresponding to an eigenvalue of the equation system) multiplied by the beam's length, if this product is great enough, its exponential can't be calculated in the range of a classical machine precision, a solution would be to discretize sufficiently the beam to reduce the length of the beam elements, or to eliminate the corresponding mode associated to this eigenvalue.

APPENDIX A1: Stiffness matrices for the triangular and the 1D element.

For the 1D element, we use the stiffness matrix of a planar Euler-Bernoulli element, expressed by:

$$\boldsymbol{K}_b = \begin{bmatrix} \frac{ES}{l} & 0 & 0 & -\frac{ES}{l} & 0 & 0 \\ & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ & & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ & & & \frac{ES}{l} & 0 & 0 \\ & sym & & & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ & & & & & \frac{4EI}{l} \end{bmatrix} \quad \text{a.1}$$

Where $S = t^2$ is the surface, t the thin walled profile thickness, l its length and $I = t^4/12$ its inertia.

For the triangular element, we use the rigidity matrix in planar stress, expressed by:

$$\boldsymbol{K}_t = \frac{Et}{4\Delta(1-\upsilon^2)} \begin{bmatrix} a_1^2 + \alpha b_1^2 & (\alpha+\upsilon)a_1 b_1 & a_1 a_2 + \alpha b_1 b_2 & \upsilon a_1 b_2 + \alpha a_2 b_1 & a_1 a_3 + \alpha b_1 b_3 & \upsilon a_1 b_3 + \alpha a_3 b_1 \\ & b_1^2 + \alpha a_1^2 & \upsilon a_2 b_1 + \lambda a_1 b_2 & b_1 b_2 + \alpha a_1 a_2 & \upsilon a_3 b_1 + \alpha a_1 b_3 & a_1 a_3 + \alpha b_1 b_3 \\ & & a_2^2 + \alpha b_2^2 & (\upsilon+\alpha) a_2 b_2 & a_2 a_3 + \alpha b_2 b_3 & \upsilon a_2 b_3 + \alpha a_3 b_2 \\ & & & b_2^2 + \alpha a_2^2 & \upsilon a_3 b_2 + \alpha a_2 b_3 & a_2 a_3 + \alpha b_2 b_3 \\ & sym & & & a_3^2 + \alpha b_3^2 & (\upsilon+\alpha) a_3 b_3 \\ & & & & & b_3^2 + \alpha a_3^2 \end{bmatrix} \quad \text{a.2}$$

Where Δ is the triangular element surface, t its thickness and $\alpha = (1 - \upsilon)/2$.

And for (x_1, y_1), (x_2, y_2), (x_3, y_3) the triangular element nodes coordinate we have:

$$\begin{array}{lll} a_1 = y_2 - y_3 & a_2 = y_3 - y_1 & a_3 = y_1 - y_2 \\ b_1 = x_3 - x_2 & b_2 = x_1 - x_3 & b_3 = x_2 - x_1 \end{array}$$

APPENDIX A2 : derivation of the warping functions for thin walled profiles.

For thin walled profiles we make two approximations:

- The tangential shear stress in the cross section is tangential to the thin walled profile.
- The tangential shear stress is constant in the thin walled profile thickness.

The expression of the tangential shear stress vector is given by:

$$\boldsymbol{\tau} = \mu(\boldsymbol{\psi} - \boldsymbol{\nabla}\Omega)\frac{d\zeta}{dx}$$

$$\tau_s = \boldsymbol{\tau} \cdot \boldsymbol{t}$$

$$\tau_s = \mu(\boldsymbol{\psi} \cdot \boldsymbol{t} - \boldsymbol{\nabla}\Omega \cdot \boldsymbol{t})\frac{d\zeta}{dx} \qquad \Rightarrow \qquad \frac{\partial\Omega}{\partial s} = \boldsymbol{\psi} \cdot \boldsymbol{t} - \frac{\tau_s}{\mu\frac{d\zeta}{dx}} \tag{a.3}$$

Where $\boldsymbol{t}$ is the tangential vector to the thin walled profile, and $\boldsymbol{\nabla}\Omega \cdot \boldsymbol{t} = \frac{\partial\Omega}{\partial s}$

For a section free to warp, we have as already written:

$$\frac{\partial u_p}{\partial x} = 0 \quad \Rightarrow \quad \Omega\frac{d^2\zeta}{dx^2} = 0 \quad \Rightarrow \quad \frac{d\zeta}{dx} = cst$$

We consider then that $\mu\frac{d\zeta}{dx} = \frac{1}{J}$, with J a constant. We can then write:

$$\frac{\partial\Omega}{\partial s} = \boldsymbol{\psi} \cdot \boldsymbol{t} - J\tau_s \qquad \Rightarrow \qquad \frac{\partial^2\Omega}{\partial s^2} = \frac{\partial}{\partial s}(\boldsymbol{\psi} \cdot \boldsymbol{t}) - J\frac{\partial\tau_s}{\partial s} \tag{a.4}$$

We write the equilibrium equation of the beam:

$$\frac{\partial\sigma_{xx}}{\partial x} + div\ \boldsymbol{\tau} = \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_s}{\partial s} = 0$$

$$J\frac{\partial\tau_s}{\partial s} = -\frac{\lambda}{\mu}\ div\ \boldsymbol{\psi} \tag{a.5}$$

After replacing this in the equation a.2, we obtain :

$$\frac{\partial^2 \Omega}{\partial s^2} = \frac{\partial}{\partial s}(\boldsymbol{\psi} \cdot \boldsymbol{t}) + \frac{\lambda}{\mu}\, div\, \boldsymbol{\psi} \qquad \text{a.6}$$

By integrating this equation twice, we obtain Ω as a function of its limit values in the extremities of the thin walled profile. After assembling the equations expressing Ω for each thin profile constituting the section, we obtain a system of equations that can be solved to an additive constant, thus to solve the problem uniquely we add the condition $\int_A \Omega\, dA = 0$.

APPENDIX A3 : proof of proposition 3.

We replace the obtained solution $\boldsymbol{R}\boldsymbol{e}^{\boldsymbol{S}x}\boldsymbol{a}$ in the equation system:

$$(\boldsymbol{TRS}^2 + \boldsymbol{URS} - \boldsymbol{VR})\boldsymbol{e}^{\boldsymbol{S}x}\boldsymbol{a} = \boldsymbol{0} \qquad \text{a.7}$$

This relation is valid for an arbitrary vector $\boldsymbol{a}$, thus:

$$\boldsymbol{TRS}^2 + \boldsymbol{URS} - \boldsymbol{VR} = \boldsymbol{0} \qquad \text{a.8}$$

By developing this relation we obtain:

$$\begin{aligned}&\left(\boldsymbol{T}\left(\boldsymbol{R}_r(\boldsymbol{S}_r^2 - \boldsymbol{S}_i^2) - 2\boldsymbol{R}_i\boldsymbol{S}_i\boldsymbol{S}_r\right) + \boldsymbol{U}(\boldsymbol{R}_r\boldsymbol{S}_r - \boldsymbol{R}_i\boldsymbol{S}_i) - \boldsymbol{V}\boldsymbol{R}_r\right)\\ &\qquad + i\left(\boldsymbol{T}\left(\boldsymbol{R}_i(\boldsymbol{S}_r^2 - \boldsymbol{S}_i^2) + 2\boldsymbol{R}_r\boldsymbol{S}_i\boldsymbol{S}_r\right) + \boldsymbol{U}(\boldsymbol{R}_r\boldsymbol{S}_i + \boldsymbol{R}_i\boldsymbol{S}_r) - \boldsymbol{V}\boldsymbol{R}_i\right) = \boldsymbol{0}\end{aligned} \qquad \text{a.9}$$

Thus, we have the following two formulas:

$$\boldsymbol{T}\left(\boldsymbol{R}_r(\boldsymbol{S}_r^2 - \boldsymbol{S}_i^2) - 2\boldsymbol{R}_i\boldsymbol{S}_i\boldsymbol{S}_r\right) + \boldsymbol{U}(\boldsymbol{R}_r\boldsymbol{S}_r - \boldsymbol{R}_i\boldsymbol{S}_i) - \boldsymbol{V}\boldsymbol{R}_r = \boldsymbol{0} \qquad \text{a.10}$$

$$\boldsymbol{T}\left(\boldsymbol{R}_i(\boldsymbol{S}_r^2 - \boldsymbol{S}_i^2) + 2\boldsymbol{R}_r\boldsymbol{S}_i\boldsymbol{S}_r\right) + \boldsymbol{U}(\boldsymbol{R}_r\boldsymbol{S}_i + \boldsymbol{R}_i\boldsymbol{S}_r) - \boldsymbol{V}\boldsymbol{R}_i = \boldsymbol{0} \qquad \text{a.11}$$

We calculate now the first and second derivate of $\boldsymbol{\phi}_x = (\boldsymbol{R}_r\boldsymbol{Y} + \boldsymbol{R}_i\boldsymbol{Z})\boldsymbol{e}^{\boldsymbol{S}_r x}\boldsymbol{a}$:

$$\boldsymbol{\phi}_x{}' = (\boldsymbol{R}_r\boldsymbol{S}_r\boldsymbol{Y} + \boldsymbol{R}_r\boldsymbol{S}_i\boldsymbol{Z} + \boldsymbol{R}_i\boldsymbol{S}_r\boldsymbol{Z} - \boldsymbol{R}_i\boldsymbol{S}_i\boldsymbol{Y})\boldsymbol{e}^{\boldsymbol{S}_r x}\boldsymbol{a}$$

$$\boldsymbol{\phi}_x{}'' = (\boldsymbol{R}_r(\boldsymbol{S}_r^2 - \boldsymbol{S}_i^2)\boldsymbol{Y} + 2\boldsymbol{R}_r\boldsymbol{S}_i\boldsymbol{S}_r\boldsymbol{Z} + \boldsymbol{R}_i(\boldsymbol{S}_r^2 - \boldsymbol{S}_i^2)\boldsymbol{Z} - 2\boldsymbol{R}_i\boldsymbol{S}_i\boldsymbol{S}_r\boldsymbol{Y})\boldsymbol{e}^{\boldsymbol{S}_r x}\boldsymbol{a}$$

We replace the expressions of the derivatives in the equation system (62) to obtain:

$$\boldsymbol{T\phi_x}'' + \boldsymbol{U\phi_x}' - \boldsymbol{V\phi_x}$$
$$= \left(\boldsymbol{T}\left(\boldsymbol{R_r}\left(\boldsymbol{S_r^2} - \boldsymbol{S_i^2}\right) - 2\boldsymbol{R_iS_iS_r}\right) + \boldsymbol{U}(\boldsymbol{R_rS_r} - \boldsymbol{R_iS_i}) - \boldsymbol{VR_r}\right)\boldsymbol{e^{S_rx}Ya} \quad \text{a.12}$$
$$+ \left(\boldsymbol{T}\left(\boldsymbol{R_i}\left(\boldsymbol{S_r^2} - \boldsymbol{S_i^2}\right) + 2\boldsymbol{R_rS_iS_r}\right) + \boldsymbol{U}(\boldsymbol{R_rS_i} + \boldsymbol{R_iS_r}) - \boldsymbol{VR_i}\right)\boldsymbol{e^{S_rx}Za}$$

Thus, from the equations a.10 and a.11 that we replace in a.12, we obtain $\boldsymbol{T\phi_x}'' + \boldsymbol{U\phi_x}' - \boldsymbol{V\phi_x} = \boldsymbol{0}$, which verify that $\boldsymbol{\phi_x} = (\boldsymbol{R_rY} + \boldsymbol{R_iZ})\boldsymbol{e^{S_rx}a}$ is a solution of the system.

APPENDIX A4 : proof of proposition 4.

We suppose that the 2nd member $\boldsymbol{f}$ is of the form $\boldsymbol{f_x} = \boldsymbol{f_n}x^n$, we suppose then that the particular solution will be given by:

$$\boldsymbol{\phi_p} = -\sum_{i=0}^{n} \boldsymbol{RS^{-i-1}L}\,\boldsymbol{f_x}^{(i)} = -\sum_{i=0}^{n} \boldsymbol{RS^{-i-1}L}\,\boldsymbol{f_n}x^{n-i}\frac{n!}{(n-i)!} \quad \text{a.13}$$

We calculate the expression of $\boldsymbol{V\phi_p}$, $\boldsymbol{U\phi'_p}$ and $\boldsymbol{T\phi''_p}$:

$$\boldsymbol{V\phi_p} = -\sum_{i=0}^{n} \boldsymbol{VRS^{-i-1}L}\,\boldsymbol{f_n}x^{n-i}\frac{n!}{(n-i)!} = -\boldsymbol{VRS^{-1}Lf_n}x^n - \boldsymbol{VRS^{-2}Lf_n}nx^{n-1} - \sum_{i=2}^{n} \boldsymbol{VRS^{-i-1}L}\,\boldsymbol{f_n}x^{n-i}\frac{n!}{(n-i)!}$$

$$\boldsymbol{U\phi'_p} = -\sum_{i=0}^{n-1} \boldsymbol{URS^{-i-1}L}\,\boldsymbol{f_n}x^{n-i-1}\frac{n!}{(n-i-1)!} = -\boldsymbol{URS^{-1}L}\,\boldsymbol{f_n}nx^{n-1} - \sum_{i=2}^{n} \boldsymbol{URS^{-i}L}\,\boldsymbol{f_n}x^{n-i}\frac{n!}{(n-i)!}$$

$$\boldsymbol{T\phi''_p} = -\sum_{i=0}^{n-2} \boldsymbol{TRS^{-i-1}L}\,\boldsymbol{f_n}x^{n-i-2}\frac{n!}{(n-i-2)!} = -\sum_{i=2}^{n} \boldsymbol{TRS^{-i+1}L}\,\boldsymbol{f_n}x^{n-i}\frac{n!}{(n-i)!}$$

Thus :

$$\boldsymbol{T\phi''_p} + \boldsymbol{U\phi'_p} - \boldsymbol{V\phi_p}$$
$$= \boldsymbol{VRS^{-1}Lf_n}x^n - (\boldsymbol{URS} - \boldsymbol{VR})\boldsymbol{S^{-2}Lf_n}nx^{n-1} \quad \text{a.14}$$
$$-\sum_{i=2}^{n}(\boldsymbol{TRS^2} + \boldsymbol{URS} - \boldsymbol{VR})\boldsymbol{S^{-i-1}L}\,\boldsymbol{f_n}x^{n-i}\frac{n!}{(n-i)!}$$

By using the following relations deduced from the equations (75):

$$\boldsymbol{TRS^2} + \boldsymbol{URS} - \boldsymbol{VR} = \boldsymbol{0} \quad , \quad (\boldsymbol{URS} - \boldsymbol{VR})\boldsymbol{S^{-2}L} = \boldsymbol{TRL} = \boldsymbol{0} \quad , \quad \boldsymbol{RS^{-1}L} = \boldsymbol{V^{-1}}$$

We verify that: $\boldsymbol{T\phi''}_p + \boldsymbol{U\phi'}_p - \boldsymbol{V\phi}_p = \boldsymbol{f}_n x^n$, thus $\boldsymbol{\phi}_p$ is a particular solution system, and by the superposition principle, its straightforward that for a 2nd member in the differential equation system of the form $\boldsymbol{f}_x = \sum_{i=0}^{n} \boldsymbol{f}_i x^i$ the particular solution is $\boldsymbol{\phi}_p = -\sum_{i=0}^{n} \boldsymbol{R}\boldsymbol{S}^{-i-1}\boldsymbol{L}\,\boldsymbol{f}_x^{(i)}$.

APPENDIX A5 :

In the calculation of the integrals and the derivative of $\boldsymbol{\phi}_x$ expressed in the relations (85), (86) and (87), the only difficulty is the calculation of the integrals and derivative of $\boldsymbol{W}_x = (\boldsymbol{R}_r\boldsymbol{Y}_x + \boldsymbol{R}_i\boldsymbol{Z}_x)\boldsymbol{e}^{\boldsymbol{S}_r x}$ used in its expression. Thus, we want to calculate the expression of $\boldsymbol{W}_{-1x}$, $\boldsymbol{W}_{-2x}$, $\boldsymbol{W}_{-3x}$ and $\boldsymbol{W}_{1x}$:

$$\boldsymbol{W}_{-1x} = \int_0^x \boldsymbol{W}_t dt \quad , \quad \boldsymbol{W}_{-2x} = \int_0^x \boldsymbol{W}_{-1t} dt \quad , \quad \boldsymbol{W}_{-3x} = \int_0^x \boldsymbol{W}_{-2t} dt \quad , \quad \boldsymbol{W}_{1x} = \boldsymbol{W'}_x \qquad \text{a.15}$$

We have the expression of $\boldsymbol{W}_x$:

$$\boldsymbol{W}_x = (\boldsymbol{R}_r\boldsymbol{Y}_x + \boldsymbol{R}_i\boldsymbol{Z}_x)\boldsymbol{e}^{\boldsymbol{S}_r x}$$

$$= \left(\boldsymbol{R}_r \begin{bmatrix} e^{\boldsymbol{S}_1 x} & & & 0 \\ & e^{-\boldsymbol{S}_1 x} & & \\ & & \boldsymbol{X}_{sx} e^{\boldsymbol{S}_{2r} x} & \\ 0 & & & \boldsymbol{X}_{cx} e^{\boldsymbol{S}_{2r} x} \end{bmatrix} + \boldsymbol{R}_i \begin{bmatrix} e^{\boldsymbol{S}_1 x} & & & 0 \\ & e^{-\boldsymbol{S}_1 x} & & \\ & & \boldsymbol{X}_{cx} e^{\boldsymbol{S}_{2r} x} & \\ 0 & & & \boldsymbol{X}_{sx} e^{\boldsymbol{S}_{2r} x} \end{bmatrix} \right) \qquad \text{a.16}$$

If we note $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{0} & & & \boldsymbol{0} \\ & \boldsymbol{0} & & \\ & & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$ and $\boldsymbol{R}_{ri} = \boldsymbol{R}_r + \boldsymbol{R}_i\boldsymbol{Q}$ then we can express $\boldsymbol{W}_x$ in the following form :

$$\boldsymbol{W}_x = \boldsymbol{R}_{ri} \begin{bmatrix} e^{\boldsymbol{S}_1 x} & & & 0 \\ & e^{-\boldsymbol{S}_1 x} & & \\ & & \boldsymbol{X}_{sx} e^{\boldsymbol{S}_{2r} x} & \\ 0 & & & \boldsymbol{X}_{cx} e^{\boldsymbol{S}_{2r} x} \end{bmatrix} \qquad \text{a.17}$$

The 1st integral of $\boldsymbol{W}_x$ will be expressed by :

$$\boldsymbol{W}_{-1x} = \boldsymbol{R}_{ri} \begin{bmatrix} \int_0^x e^{\boldsymbol{S}_1 t} dt & & & 0 \\ & \int_0^x e^{-\boldsymbol{S}_1 t} dt & & \\ & & \int_0^x \boldsymbol{X}_{st} e^{\boldsymbol{S}_{2r} t} dt & \\ 0 & & & \int_0^x \boldsymbol{X}_{ct} e^{\boldsymbol{S}_{2r} t} dt \end{bmatrix} \qquad \text{a.18}$$

We need to calculate: $\int_0^x \boldsymbol{X}_{ct} e^{\boldsymbol{S}_{2r}t} dt$ and $\int_0^x \boldsymbol{X}_{st} e^{\boldsymbol{S}_{2r}t} dt$. This is performed easily by using the complex number:

$$\int_0^x \boldsymbol{X}_{ct} e^{\boldsymbol{S}_{2r}t} dt + i\int_0^x \boldsymbol{X}_{st} e^{\boldsymbol{S}_{2r}t} dt = \int_0^x e^{(\boldsymbol{S}_{2r}+i\boldsymbol{S}_{2i})t} dt = (\boldsymbol{S}_{2r} + i\boldsymbol{S}_{2i})^{-1}\left(e^{(\boldsymbol{S}_{2r}+i\boldsymbol{S}_{2i})x} - \boldsymbol{I}\right)$$

$$= (\boldsymbol{S}_{2r}^2 + \boldsymbol{S}_{2i}^2)^{-1}(\boldsymbol{S}_{2r} - i\boldsymbol{S}_{2i})\left((\boldsymbol{X}_{cx} + i\boldsymbol{X}_{sx})e^{\boldsymbol{S}_{2r}x} - \boldsymbol{I}\right)$$

$$= \left(\left((\boldsymbol{A}_{2r}\boldsymbol{X}_{cx} + \boldsymbol{A}_{2i}\boldsymbol{X}_{sx})e^{\boldsymbol{S}_{2r}x} - \boldsymbol{A}_{2r}\right) + i\left((\boldsymbol{A}_{2r}\boldsymbol{X}_{sx} - \boldsymbol{A}_{2i}\boldsymbol{X}_{cx})e^{\boldsymbol{S}_{2r}x} + \boldsymbol{A}_{2i}\right)\right)$$

Where : $\boldsymbol{A}_{2r} = (\boldsymbol{S}_{2r}^2 + \boldsymbol{S}_{2i}^2)^{-1}\boldsymbol{S}_{2r}$, $\boldsymbol{A}_{2r} = (\boldsymbol{S}_{2r}^2 + \boldsymbol{S}_{2i}^2)^{-1}\boldsymbol{S}_{2i}$.

Thus :

$$\int_0^x \boldsymbol{X}_{ct} e^{\boldsymbol{S}_{2r}t} dt = (\boldsymbol{A}_{2r}\boldsymbol{X}_{cx} + \boldsymbol{A}_{2i}\boldsymbol{X}_{sx})e^{\boldsymbol{S}_{2r}x} - \boldsymbol{A}_{2r} \quad \text{a.19}$$

$$\int_0^x \boldsymbol{X}_{st} e^{\boldsymbol{S}_{2r}t} dt = (\boldsymbol{A}_{2r}\boldsymbol{X}_{sx} - \boldsymbol{A}_{2i}\boldsymbol{X}_{cx})e^{\boldsymbol{S}_{2r}x} + \boldsymbol{A}_{2i} \quad \text{a.20}$$

With a.19 and a.20 we can then determine $\boldsymbol{W}_{-1x}$ easily.

In the same way, for the calculation of $\boldsymbol{W}_{-2x}$ we will need to calculate $\int_0^x \left((\boldsymbol{A}_{2r}\boldsymbol{X}_{ct} + \boldsymbol{A}_{2i}\boldsymbol{X}_{st})e^{\boldsymbol{S}_{2r}t} - \boldsymbol{A}_{2r}\right)dt$ and $\int_0^x \left((\boldsymbol{A}_{2r}\boldsymbol{X}_{sx} - \boldsymbol{A}_{2i}\boldsymbol{X}_{cx})e^{\boldsymbol{S}_{2r}x} + \boldsymbol{A}_{2i}\right)dt$, this can be done by using the expressions in a.19 and a.20. Thus:

$$\int_0^x \left((\boldsymbol{A}_{2r}\boldsymbol{X}_{ct} + \boldsymbol{A}_{2i}\boldsymbol{X}_{st})e^{\boldsymbol{S}_{2r}t} - \boldsymbol{A}_{2r}\right)dt$$

$$= \boldsymbol{A}_{2r}\left((\boldsymbol{A}_{2r}\boldsymbol{X}_{cx} + \boldsymbol{A}_{2i}\boldsymbol{X}_{sx})e^{\boldsymbol{S}_{2r}x} - \boldsymbol{A}_{2r}\right) + \boldsymbol{A}_{2i}\left((\boldsymbol{A}_{2r}\boldsymbol{X}_{sx} - \boldsymbol{A}_{2i}\boldsymbol{X}_{cx})e^{\boldsymbol{S}_{2r}x} + \boldsymbol{A}_{2i}\right) - \boldsymbol{A}_{2r}x \quad \text{a.21}$$

$$= \left((\boldsymbol{A}_{2r}^2 - \boldsymbol{A}_{2i}^2)\boldsymbol{X}_{cx} + 2\boldsymbol{A}_{2r}\boldsymbol{A}_{2i}\boldsymbol{X}_{sx}\right)e^{\boldsymbol{S}_{2r}x} - \boldsymbol{A}_{2r}x + \boldsymbol{A}_{2i}^2 - \boldsymbol{A}_{2r}^2$$

$$\int_0^x [(\boldsymbol{A}_{2r}\boldsymbol{X}_{st} - \boldsymbol{A}_{2i}\boldsymbol{X}_{ct})e^{\boldsymbol{S}_{2r}t} + \boldsymbol{A}_{2i}]dt$$

$$= \boldsymbol{A}_{2r}\left((\boldsymbol{A}_{2r}\boldsymbol{X}_{sx} - \boldsymbol{A}_{2i}\boldsymbol{X}_{cx})e^{\boldsymbol{S}_{2r}x} + \boldsymbol{A}_{2i}\right) - \boldsymbol{A}_{2i}\left((\boldsymbol{A}_{2r}\boldsymbol{X}_{cx} + \boldsymbol{A}_{2i}\boldsymbol{X}_{sx})e^{\boldsymbol{S}_{2r}x} - \boldsymbol{A}_{2r}\right) + \boldsymbol{A}_{2i}x \quad \text{a.22}$$

$$= \left((\boldsymbol{A}_{2r}^2 - \boldsymbol{A}_{2i}^2)\boldsymbol{X}_{sx} - 2\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}\boldsymbol{X}_{cx}\right)e^{\boldsymbol{S}_{2r}x} + \boldsymbol{A}_{2i}x + 2\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}$$

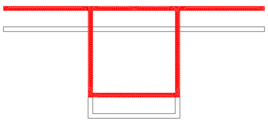
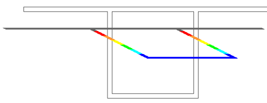
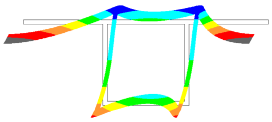
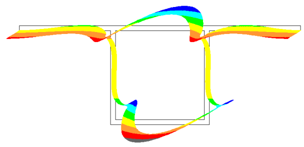
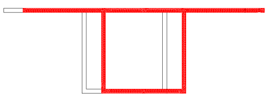
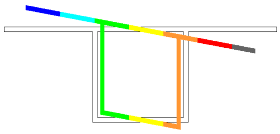
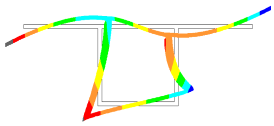
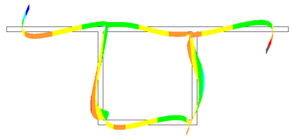
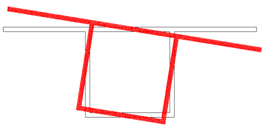
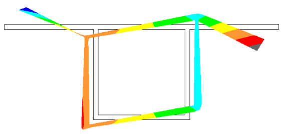
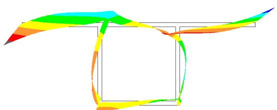
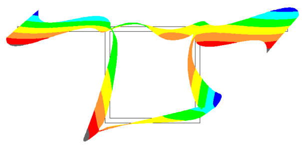
Finally for the 3rd integral, it will be expressed with the aid of the two following integrals:

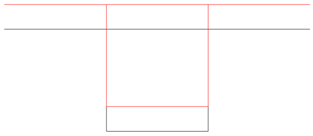
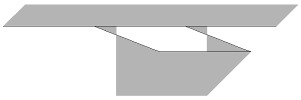
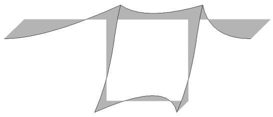
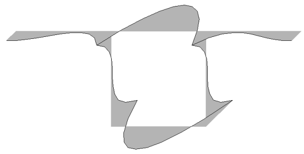
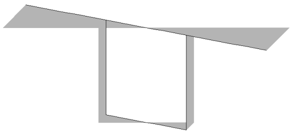
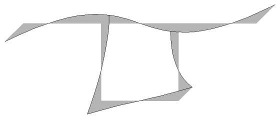
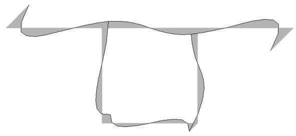
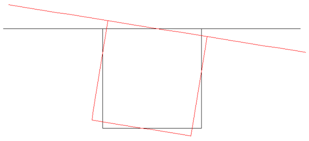
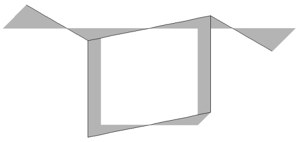
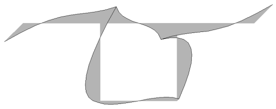
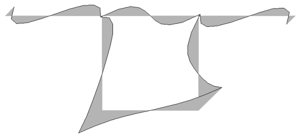
$$\int_0^x \left(\left((\boldsymbol{A}_{2r}^2 - \boldsymbol{A}_{2i}^2)\boldsymbol{X}_{ct} + 2\boldsymbol{A}_{2r}\boldsymbol{A}_{2i}\boldsymbol{X}_{st}\right)e^{S_{2r}t} - \boldsymbol{A}_{2r}t + \boldsymbol{A}_{2i}^2 - \boldsymbol{A}_{2r}^2\right)dt$$
$$= ((\boldsymbol{A}_{2r}^3 + \boldsymbol{A}_{2i}^2\boldsymbol{A}_{2r})\boldsymbol{X}_{cx} - (\boldsymbol{A}_{2i}^3 + \boldsymbol{A}_{2i}\boldsymbol{A}_{2r}^2)\boldsymbol{X}_{sx})e^{S_{2r}x} - \boldsymbol{A}_{2r}\frac{x^2}{2} + (\boldsymbol{A}_{2i}^2 - \boldsymbol{A}_{2r}^2)x - (\boldsymbol{A}_{2r}^3 - \boldsymbol{A}_{2i}^2\boldsymbol{A}_{2r}) \quad \text{a.23}$$

$$\int_0^x \left(\left((\boldsymbol{A}_{2r}^2 - \boldsymbol{A}_{2i}^2)\boldsymbol{X}_{st} - 2\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}\boldsymbol{X}_{ct}\right)e^{S_{2r}t} + \boldsymbol{A}_{2i}t + 2\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}\right)dt$$
$$= \left((\boldsymbol{A}_{2r}^3 - 3\boldsymbol{A}_{2i}^2\boldsymbol{A}_{2r})\boldsymbol{X}_{sx} + (\boldsymbol{A}_{2i}^3 - 3\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}^2)\boldsymbol{X}_{cx}\right)e^{S_{2r}x} + \boldsymbol{A}_{2i}\frac{x^2}{2} + 2\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}x + 3\boldsymbol{A}_{2i}\boldsymbol{A}_{2r}^2 - \boldsymbol{A}_{2i}^3 \quad \text{a.24}$$

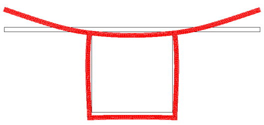
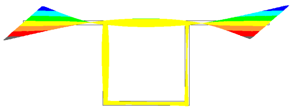
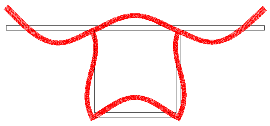
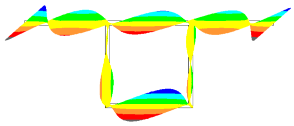
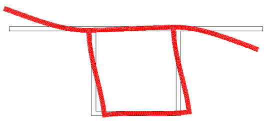
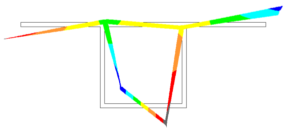
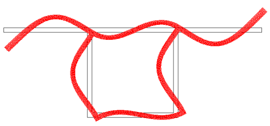
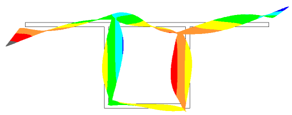
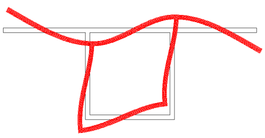
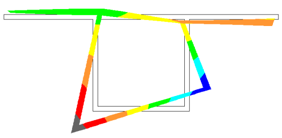
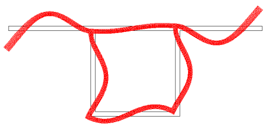
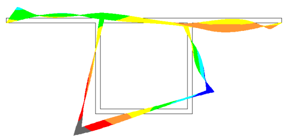
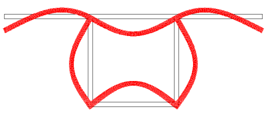
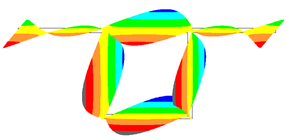
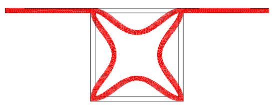
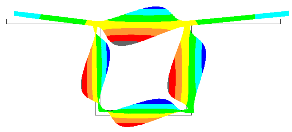
APPENDIX A6 :

Rigid body motion modes:

Transversal modes	1st warping mode	2nd warping mode	3rd warping mode
			
			
			

Transversal modes	1st warping mode	2nd warping mode	3rd warping mode
			
			
			

<u>Higher transversal modes:</u>

Transversal mode	Corresponding 1st warping mode	Transversal mode	Corresponding 1st warping mode
			
			
			
			

APPENDIX A7:

For the box girder section, we consider ten transversal modes, with one warping mode associated to each one of them. We give the numerical values of the matrices introduced in page 12. The authors are also disposed to provide to the interested reader, more numerical values for different cross section form and for any number of distortion modes with their associated warping modes.

$$\boldsymbol{C} = 10^{-5}\mu \begin{vmatrix} 29750 & & & & & & & & & \\ & 29750 & & & & & & & & \\ & & 19090 & & & & & \boldsymbol{0} & & \\ & & & 3070 & & & & & & \\ & & & & 5170 & & & & & \\ & & & & & 10410 & & & & \\ & & & & & & 9090 & & & \\ & & & & & & & 5650 & & \\ & \boldsymbol{0} & & & & & & & 6600 & \\ & & & & & & & & & 4930 \end{vmatrix}, \quad \boldsymbol{p} = \frac{\lambda\, 10^{-5}}{(2\mu+\lambda)S} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 3.43 \\ 0 \\ -0.27 \\ 3.21 \\ -73.75 \\ -2.10 \\ 3.75 \end{Bmatrix}$$

$$\boldsymbol{H} = 10^{-5}\mu \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.70 & 0.05 & 0.04 & -0.06 & 0.11 & -0.05 & -0.05 \\ 0 & 0 & 0 & 0.05 & 16.49 & 0.22 & -0.05 & 0.08 & -0.25 & -0.09 \\ 0 & 0 & 0 & 0.04 & 0.22 & 64.68 & -0.11 & 0.16 & -0.43 & -0.38 \\ 0 & 0 & 0 & -0.06 & -0.05 & -0.11 & 227.39 & -0.68 & 0.41 & 0.47 \\ 0 & 0 & 0 & 0.11 & 0.08 & 0.16 & -0.68 & 320.32 & -1.45 & -1.56 \\ 0 & 0 & 0 & -0.05 & -0.25 & -0.43 & 0.41 & -1.45 & 432.29 & 3.93 \\ 0 & 0 & 0 & -0.05 & -0.09 & -0.38 & 0.47 & -1.56 & 3.93 & 529.50 \end{vmatrix}$$

$$\boldsymbol{F} = 10^{-5}\lambda \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.13 & 0.10 & 0.09 & -0.10 & 0.11 & -0.10 & -0.11 \\ 0 & 0 & 0 & 0.10 & 35.13 & 0.25 & -0.11 & 0.19 & -0.37 & -0.19 \\ 0 & 0 & 0 & 0.09 & 0.25 & 138.22 & -0.24 & 0.32 & -0.54 & -0.79 \\ 0 & 0 & 0 & -0.10 & -0.11 & -0.24 & 484.53 & -0.99 & 0.89 & 0.99 \\ 0 & 0 & 0 & 0.11 & 0.19 & 0.32 & -0.99 & 685.20 & -3.01 & -3.28 \\ 0 & 0 & 0 & -0.10 & -0.37 & -0.54 & 0.89 & -3.01 & 924.40 & 5.97 \\ 0 & 0 & 0 & -0.11 & -0.19 & -0.79 & 0.99 & -3.28 & 5.97 & 1133.80 \end{vmatrix}$$

$$\boldsymbol{K} = 10^{-5}(2\mu+\lambda) \begin{vmatrix} 14109.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4982.74 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 228.73 & -1.18 & -204.43 & 231.24 & -0.04 & 0.89 & -43.12 & 52.20 \\ 0 & 0 & -1.18 & 0.85 & 1.28 & -1.84 & 0.06 & 0.14 & 0.21 & -0.41 \\ 0 & 0 & -204.43 & 1.28 & 237.41 & -362.59 & -0.05 & -0.53 & 34.59 & -80.86 \\ 0 & 0 & 231.24 & -1.84 & -362.59 & 701.44 & 0.23 & 0.06 & -30.16 & 154.81 \\ 0 & 0 & -0.04 & 0.06 & -0.05 & 0.23 & 2.12 & 0.07 & 0.02 & 0.05 \\ 0 & 0 & 0.89 & 0.14 & -0.53 & 0.06 & 0.07 & 1.37 & -0.20 & 0.03 \\ 0 & 0 & -43.12 & 0.21 & 34.59 & -30.16 & 0.02 & -0.20 & 10.44 & -7.12 \\ 0 & 0 & 52.20 & -0.41 & -80.86 & 154.81 & 0.05 & 0.03 & -7.12 & 35.50 \end{vmatrix}$$

$$\boldsymbol{J} = 10^{-5}\mu \begin{vmatrix} 29750.00 & 0 & -1186.52 & 9.85 & 2136.78 & -6082.40 & -8.81 & 132.19 & -6168.80 & 3112.52 \\ 0 & 29750.00 & -0.63 & -4725.72 & 56.02 & 16.07 & -1543.38 & -4272.53 & -151.85 & -139.75 \\ -1186.52 & -0.63 & 13997.50 & 17.58 & 2227.67 & 2076.11 & 4.42 & -37.87 & 1495.99 & 209.45 \\ 9.85 & -4725.72 & 17.58 & 3811.80 & -16.23 & -10.50 & 248.05 & 664.51 & 17.18 & 22.32 \\ 2136.78 & 56.02 & 2227.67 & -16.23 & 4263.84 & -1285.47 & -5.28 & 15.71 & -990.77 & 96.36 \\ -6082.40 & 16.07 & 2076.11 & -10.50 & -1285.47 & 10945.70 & -1.36 & -16.49 & 832.41 & -732.00 \\ -8.81 & -1543.38 & 4.42 & 248.05 & -5.28 & -1.36 & 9083.50 & 238.16 & 9.36 & 5.56 \\ 132.19 & -4272.53 & -37.87 & 664.51 & 15.71 & -16.49 & 238.16 & 6166.33 & 2.10 & 37.52 \\ -6168.80 & -151.85 & 1495.99 & 17.18 & -990.77 & 832.41 & 9.36 & 2.10 & 7455.76 & -753.63 \\ 3112.52 & -139.75 & 209.45 & 22.32 & 96.36 & -732.00 & 5.56 & 37.52 & -753.63 & 5127.14 \end{vmatrix}$$

$$\boldsymbol{D} = 10^{-5}\mu \begin{vmatrix} 0.00 & 29750.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -29750.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.63 & -1186.52 & 13950.30 & 17.67 & 2313.21 & 1834.02 & 4.38 & -33.31 & 1249.04 & 333.72 \\ 4725.72 & 9.85 & 17.83 & 3060.96 & -8.11 & -6.03 & 2.97 & -14.11 & -4.71 & -0.70 \\ -56.02 & 2136.78 & 2312.94 & -7.90 & 4110.37 & -848.01 & -2.01 & 14.77 & -548.58 & -128.97 \\ -16.07 & -6082.40 & 1833.55 & -6.37 & -848.81 & 9700.45 & -1.56 & 11.41 & -427.51 & -94.34 \\ 1543.38 & -8.81 & 4.06 & 2.68 & -1.69 & -2.26 & 9002.08 & 16.18 & -0.47 & -0.90 \\ 4272.53 & 132.19 & -32.73 & -14.41 & 14.21 & 12.77 & 16.88 & 5550.64 & 7.83 & 3.80 \\ 151.85 & -6168.80 & 1249.90 & -4.69 & -547.69 & -429.27 & -0.69 & 8.31 & 6176.07 & -107.71 \\ 139.75 & 3112.52 & 333.54 & -1.05 & -126.75 & -95.57 & -0.51 & 3.19 & -109.54 & 4799.30 \end{vmatrix}$$

$$\boldsymbol{Q} = 10^{-5}\mu \begin{vmatrix} 0 & 0 & 0 & -0.06 & -10.67 & 0.85 & -0.12 & 1.07 & -24.94 & -85.70 \\ 0 & 0 & 0 & 5.20 & -0.05 & -0.05 & -17.64 & -29.83 & -0.53 & -1.68 \\ 0 & 0 & 0 & -0.02 & -1.71 & -3.49 & 0.06 & -0.74 & 25.44 & 16.90 \\ 0 & 0 & 0 & -0.29 & -0.01 & -0.02 & 4.13 & -6.25 & -0.17 & 0.16 \\ 0 & 0 & 0 & 0.00 & -0.94 & -1.99 & -0.06 & 0.27 & -9.77 & 4.16 \\ 0 & 0 & 0 & 0.02 & 2.13 & 6.07 & 0.02 & -0.59 & 24.04 & 1.99 \\ 0 & 0 & 0 & 1.02 & -0.02 & -0.01 & 18.41 & 0.97 & 0.03 & -0.01 \\ 0 & 0 & 0 & 0.28 & 0.01 & 0.01 & 0.65 & 13.51 & -0.01 & 0.15 \\ 0 & 0 & 0 & 0 & -0.39 & -0.28 & 0.03 & -0.01 & 17.33 & -3.08 \\ 0 & 0 & 0 & -0.03 & -1.06 & -0.98 & 0.05 & -0.11 & 3.46 & 15.21 \end{vmatrix}$$

REFERENCES:

[1]. Bauchau, O.A. (1985). A beam theory for anisotropic materials. *Journal of Applied Mechanics, 52, 416-422.*

[2]. Sapountzakis, E.J., Mokos,V.G. (2003). Warping shear stresses in nonuniform torsion by BEM. *Computational Mechanics, 30, 131-142.*

[3]. Sapountzakis, E.J., Mokos,V.G. (2004). 3-D beam element of variable composite cross section including warping effect. *ECCOMAS(2004).*

[4]. Silvestre, N., & Camotim, D. (2003). Nonlinear generalized beam theory for cold-formed steel members. *International Journal of Structural Stability and Dynamics, 3(04), 461-490.*

[5]. Bebiano, R., Silvestre, N., & Camotim, D. (2008). GBTUL-a code for the buckling analysis of cold-formed steel members. *In Proceedings of 19th International Specialty Conference on Recent Research and Developments in Cold-Formed Steel Design and Construction (pp. 14-15).*

[6]. Ferradi, M. K., Cespedes, X., & Arquier, M. (2013). A higher order beam finite element with warping eigenmodes. *Engineering Structures*, *46*, 748-762.

[7]. Jang, G. W., Kim, M. J., & Kim, Y. Y. (2012). Analysis of Thin-Walled Straight Beams with Generally-Shaped Closed Sections Using Numerically-Determined Sectional Deformation Functions. *Journal of Structural Engineering*, *1*, 438.

[8]. Vlassov , V.Z. (1962). Pieces longues en voiles minces. *Eyrolles, Paris*.

[9]. Tisseur, F., & Meerbergen, K. (2001). The quadratic eigenvalue problem. *Siam Review*, *43*(2), 235-286.

[10]. Bathe, K.-J. (1996). Finite Element Procedures. *Prentice-Hall, Englewood Cliffs, NJ.*

[11]. Ferradi, M. K. (2013). Corotational method and differential geometry. *International Journal of Research and Reviews in Applied Sciences, 15(1), 51-56.*

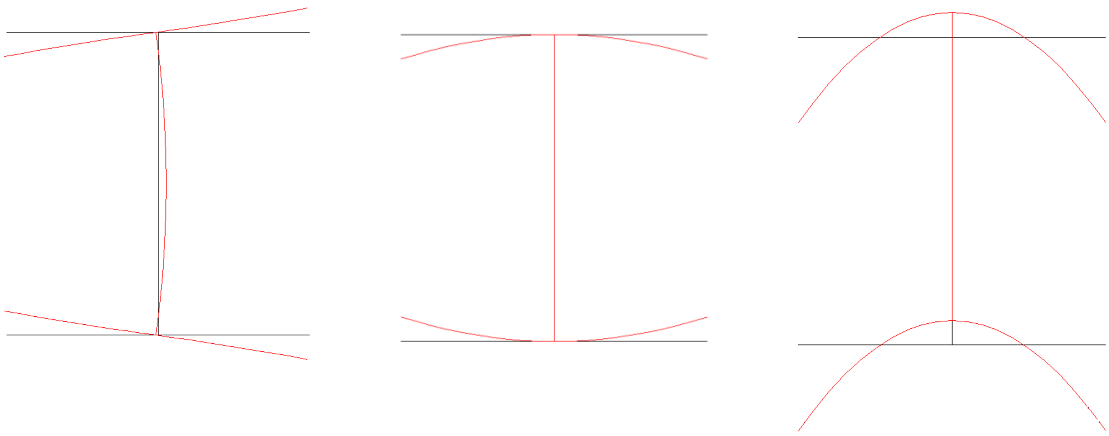


Figure 1: examples of transversal deformation modes for a thin walled profile I-section with 1D elements.

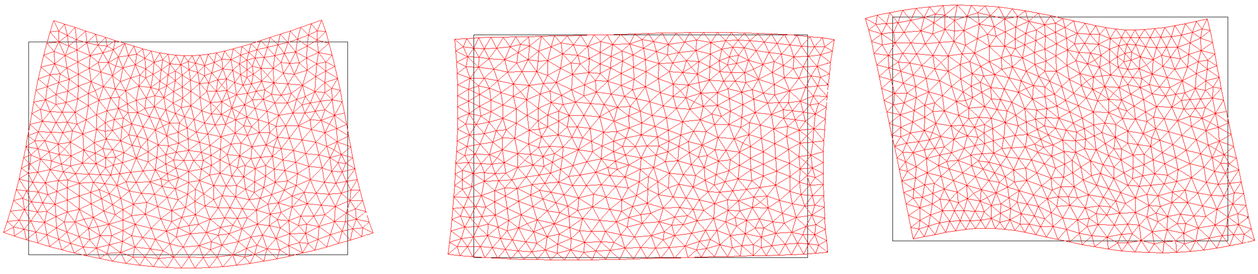


Figure 2: examples of transversal deformation modes for a rectangular section with triangular elements.

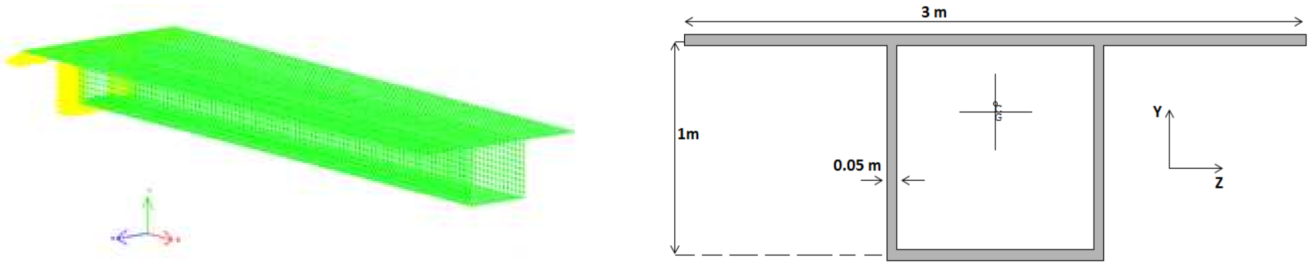


Figure 3 : A view of the shell model of the beam andits cross section.

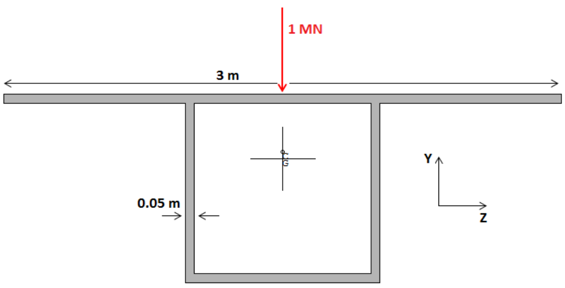


Figure 4: beam cross section with the centred applied load.

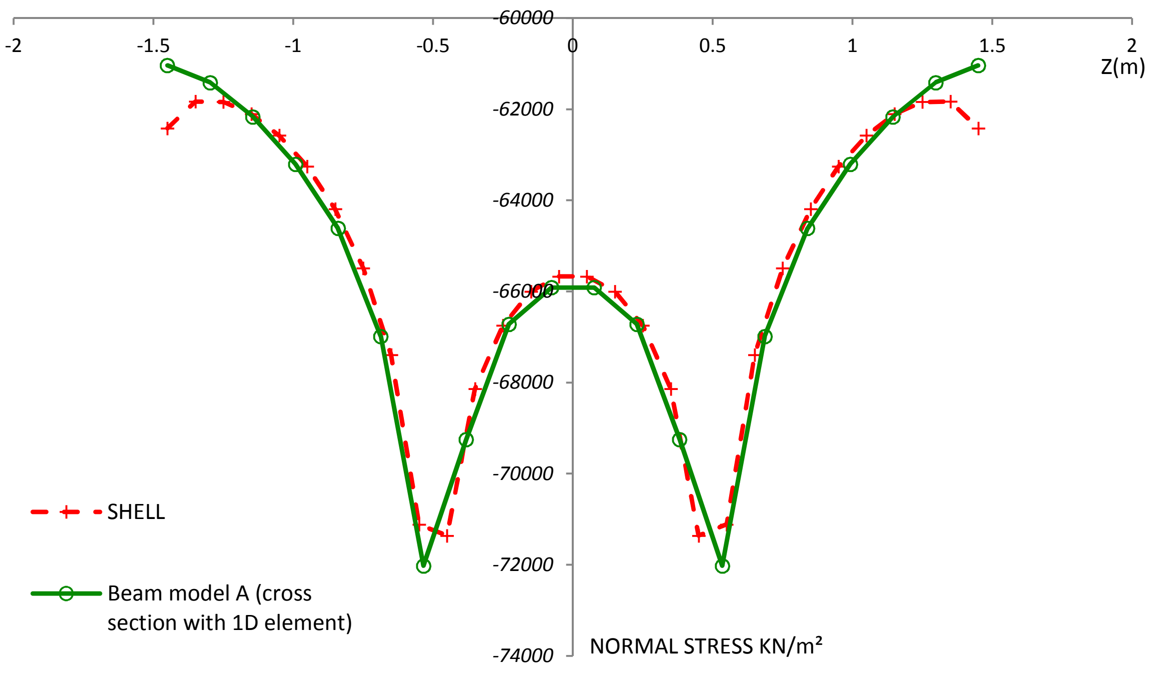


Figure 5 : Comparison of the normal stresses between the shell and the beam model, at x= 0.05m and at mid-depth of the upper slab (Ty = -1 MN)

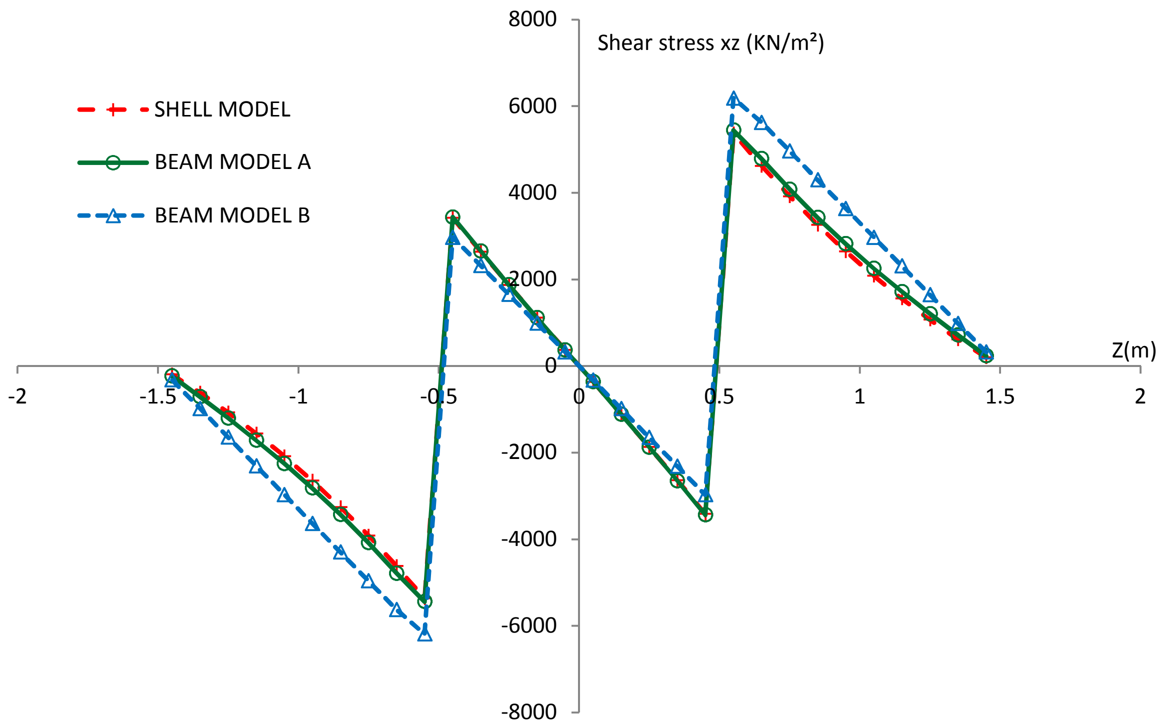


Figure 6 : Comparison of the shear stress xz between the shell and the beam models, at x= 0.95m and at mid-depth of the upper slab (Ty = -1 MN)

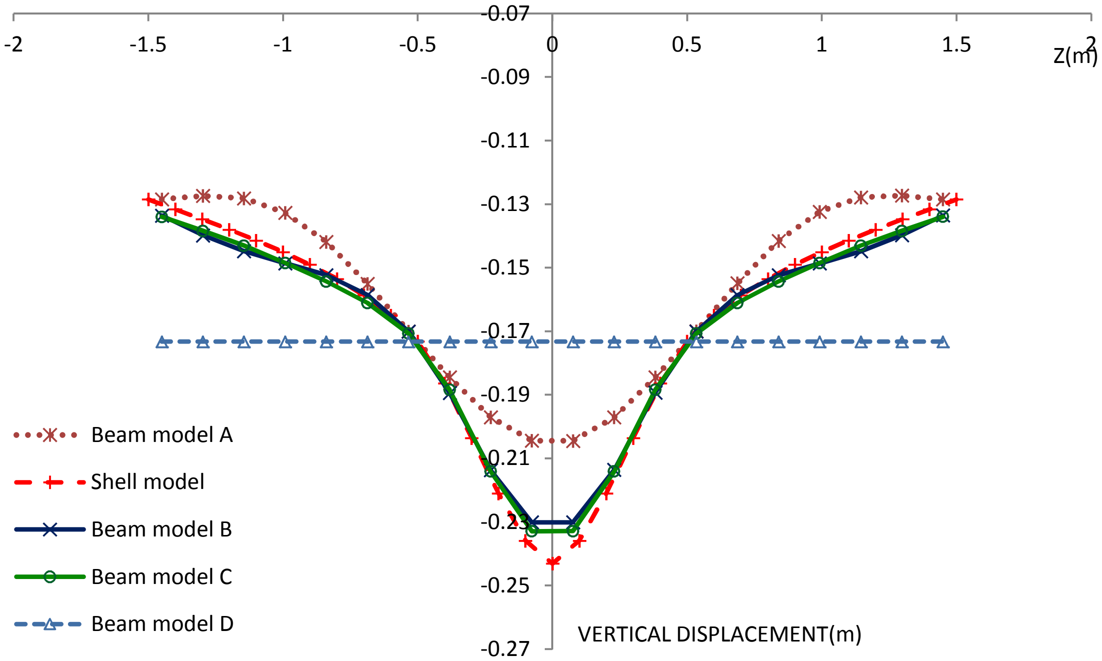


Figure 7 : Comparison of the displacement between the shell and the beam models, at x= 10m and at mid-depth of the upper slab (Ty = -1 MN)

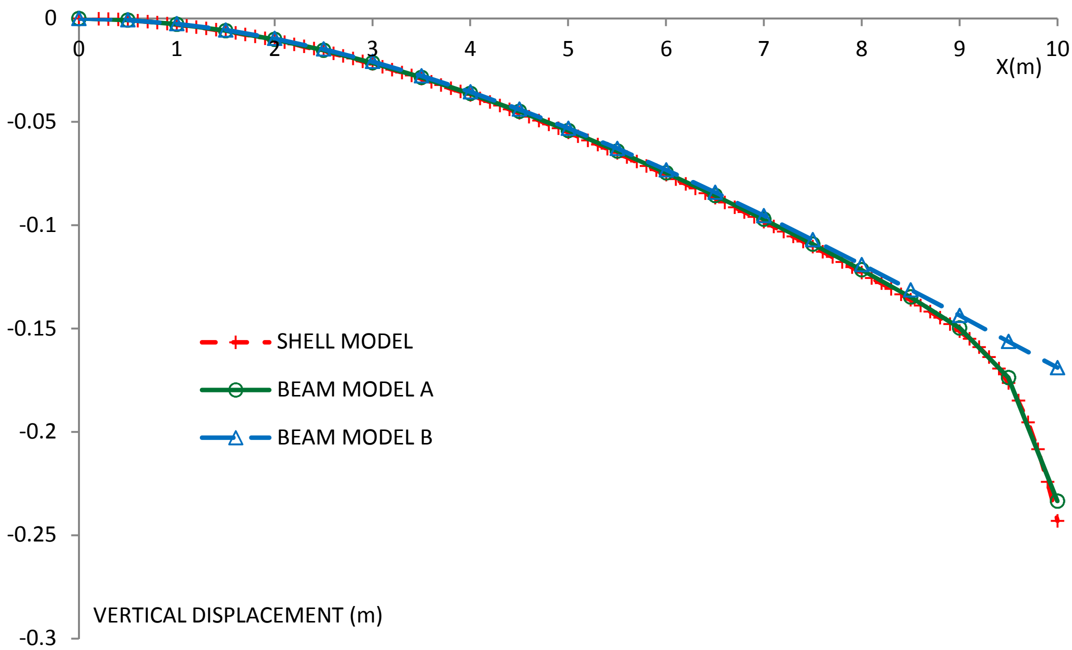


Figure 8 : Comparison of the vertical displacement along the beam length between the shell and the beam models.

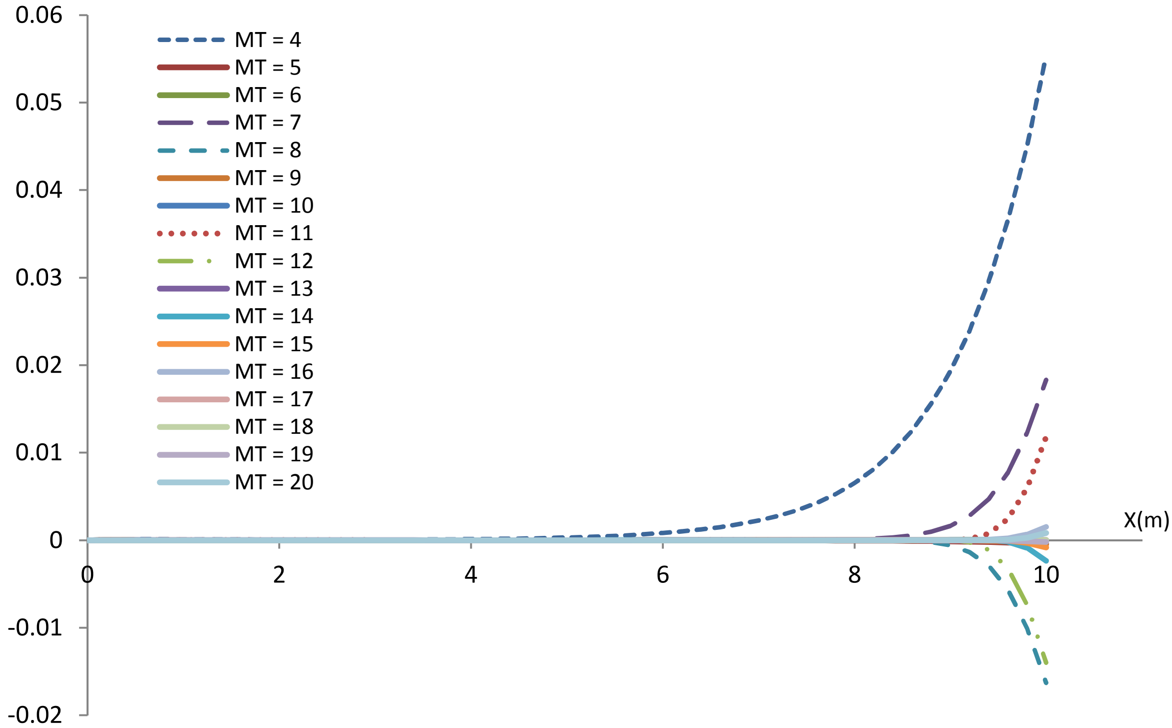


Figure 9 : representation of the generalized coordinates associated with the higher transversal modes(4-20) along the beam's length, for the beam model C in Fig.7.

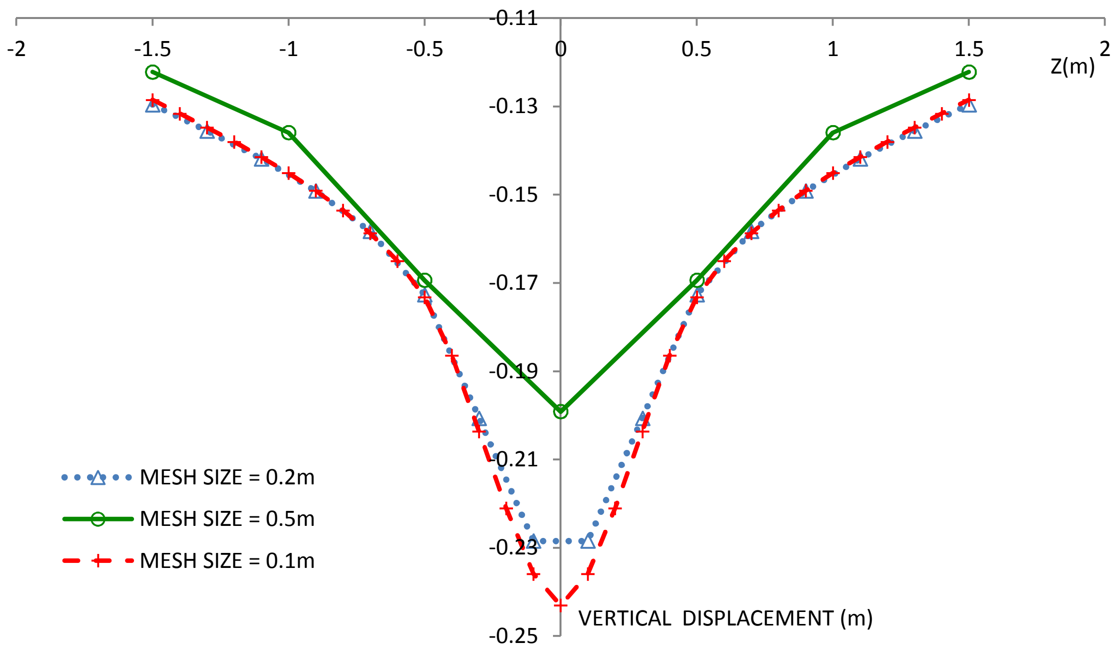


Figure 10 : Comparison of the results obtained from the shell model with different mesh size.

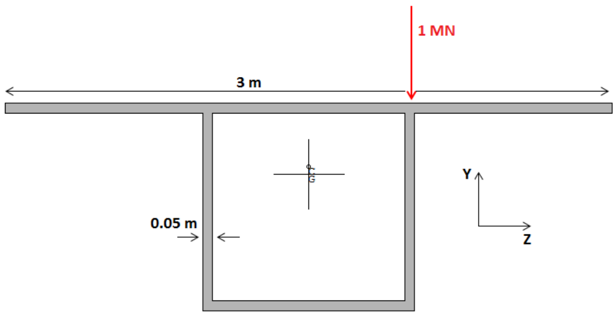


Figure 11: beam cross section with the excentred applied load.

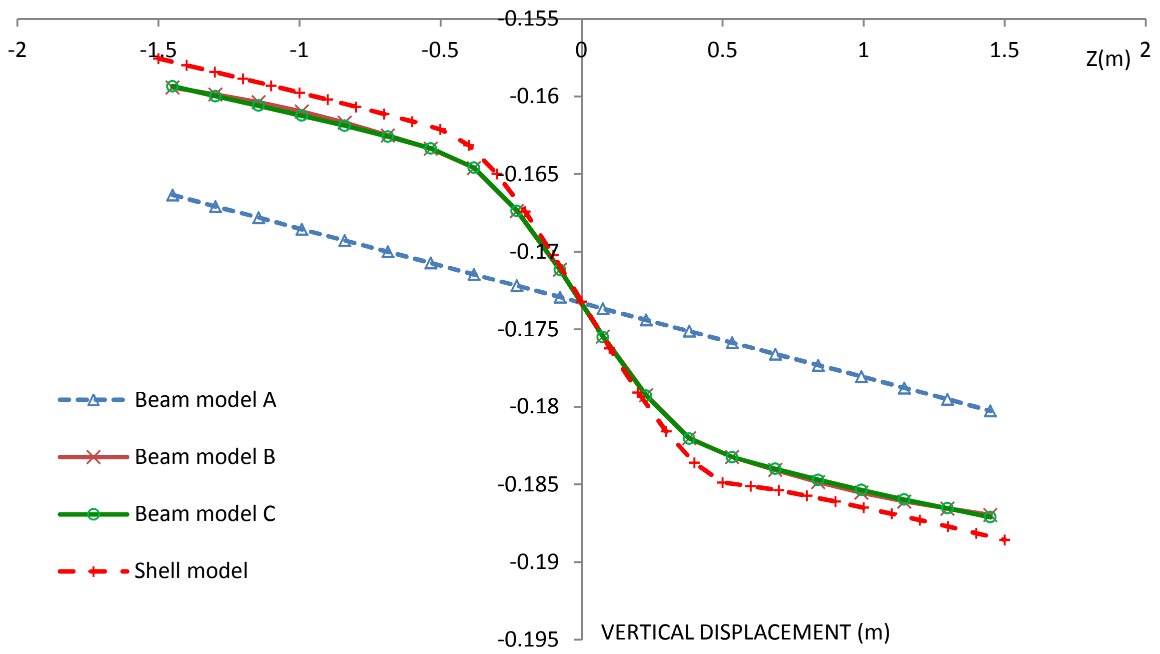


Figure 12 : Comparison of the displacement between the shell and the beam models, at x= 10m and at mid-depth of the upper slab (Ty = -1 MN)

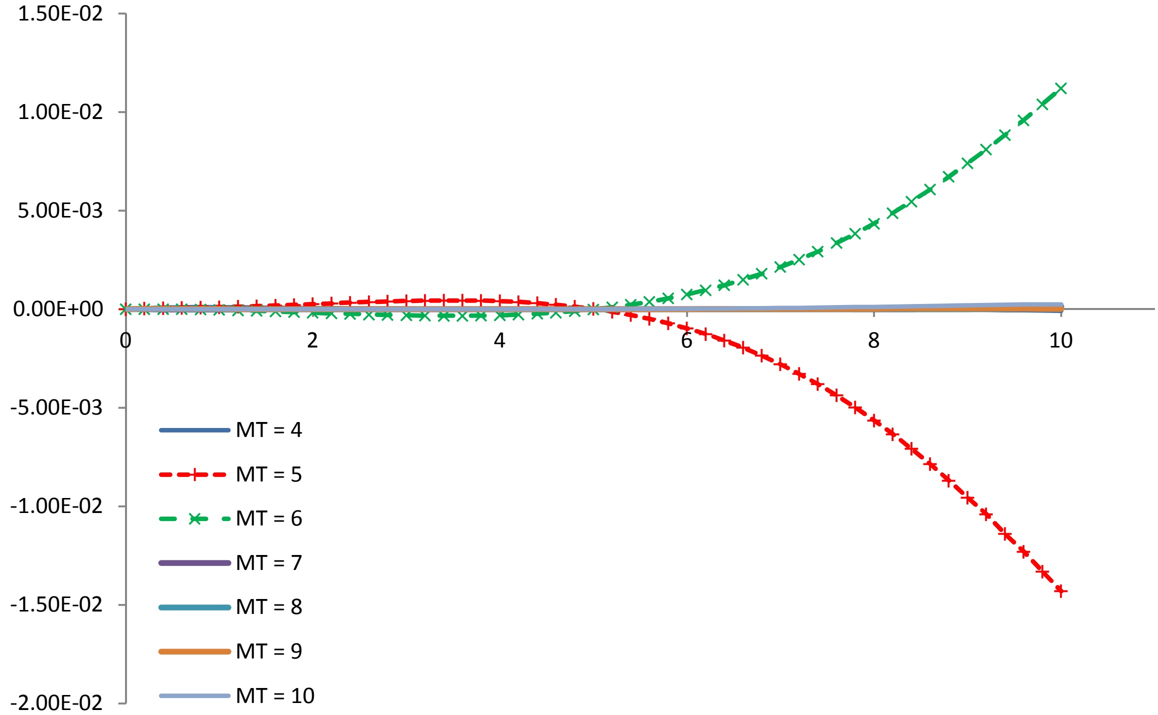


Figure 13 : representation of the generalized coordinates associated to the higher transversal modes(4-10) along the beam's length, for the beam model B in Fig.12.

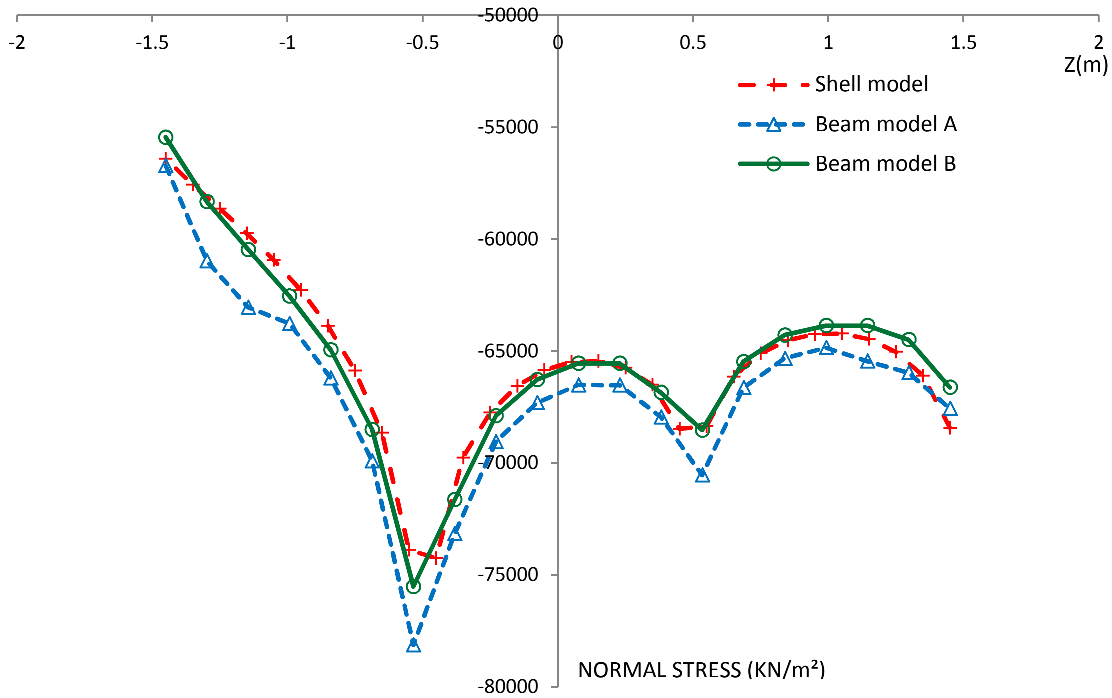


Figure 14 : Comparison of the normal stresses between the shell and the beam model, at x= 0.05m and at mid-depth of the upper slab (Ty = -1 MN)

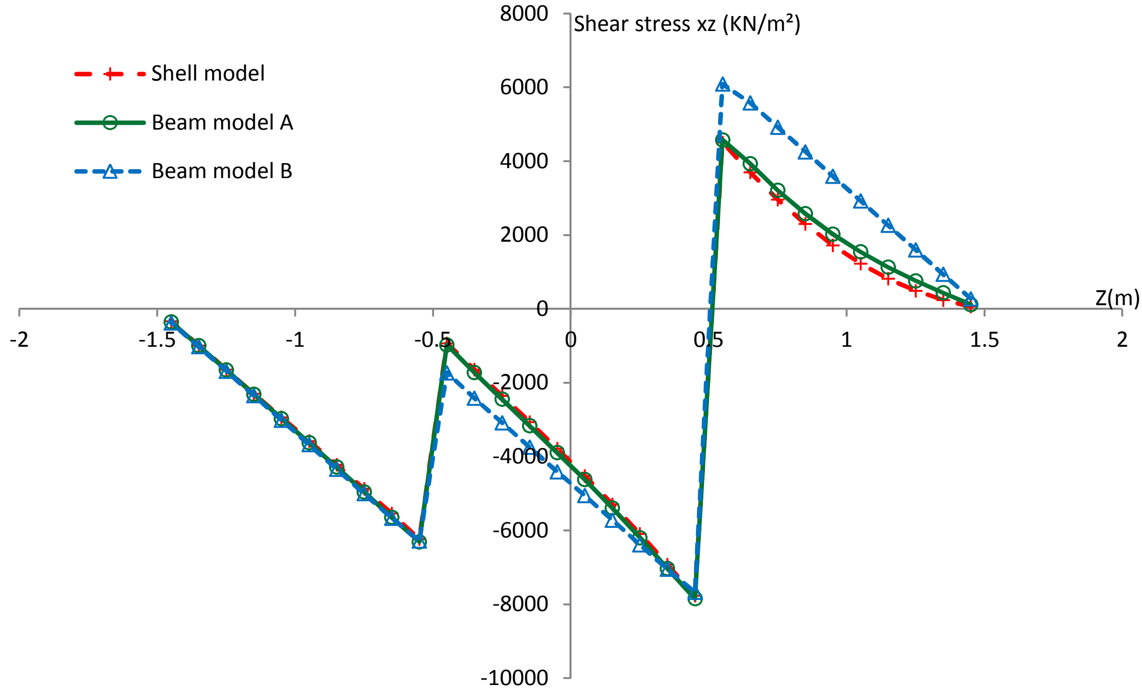


Figure 15 : Comparison of the shear stress xz between the shell and the beam models, at x= 0.95m and at mid-depth of the upper slab (Ty = -1 MN)

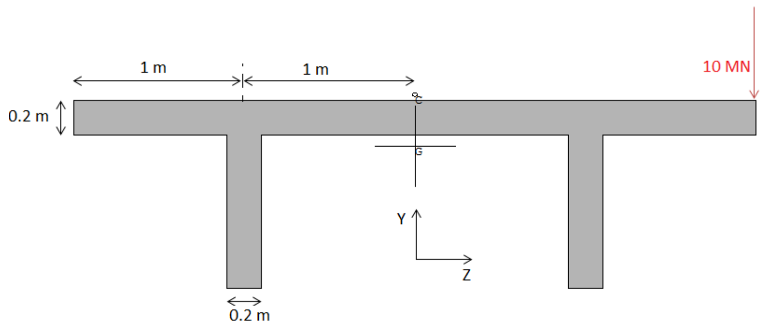


Figure 16: A view of the beam cross section.

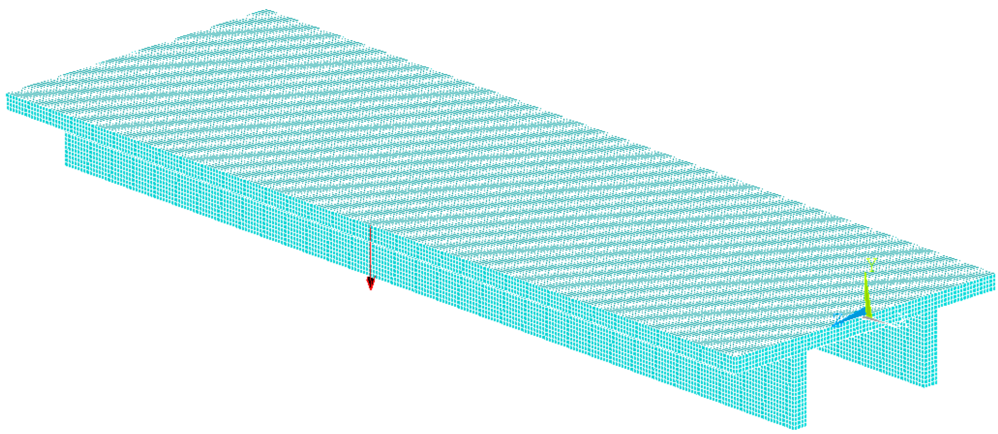


Figure 17: A view of the brick model of the beam.

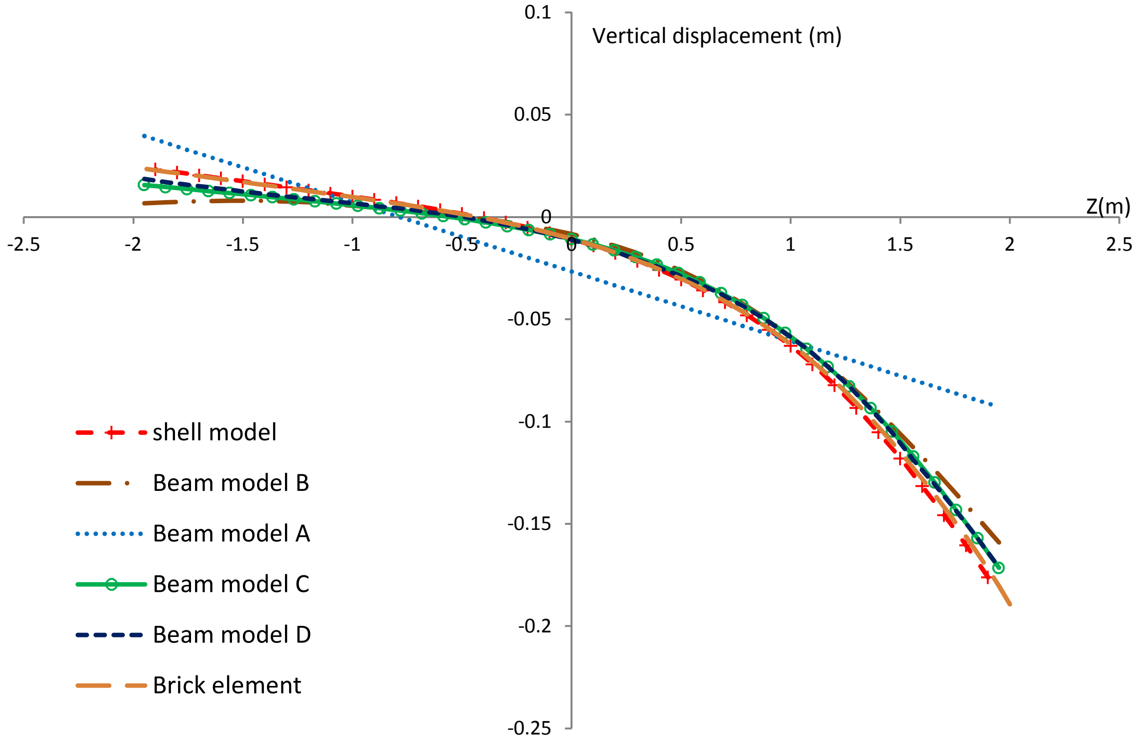


Figure 18: Comparison of the displacement between the brick, shell and the beam models, at x= 6m and at mid-depth of the upper slab (Ty = -10 MN)

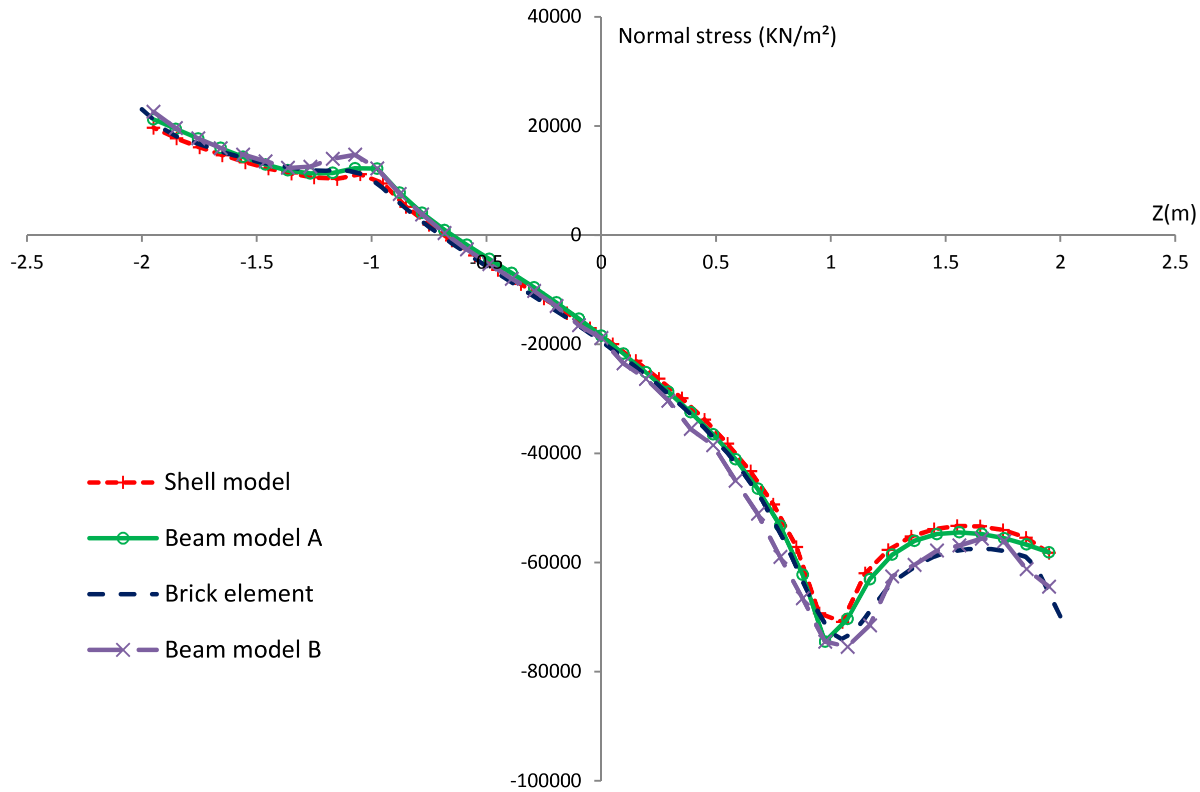


Figure 19 : Comparison of the normal stresses between the brick, shell and beam model, at x= 0.05m and at mid-depth of the upper slab (Ty = -10 MN)

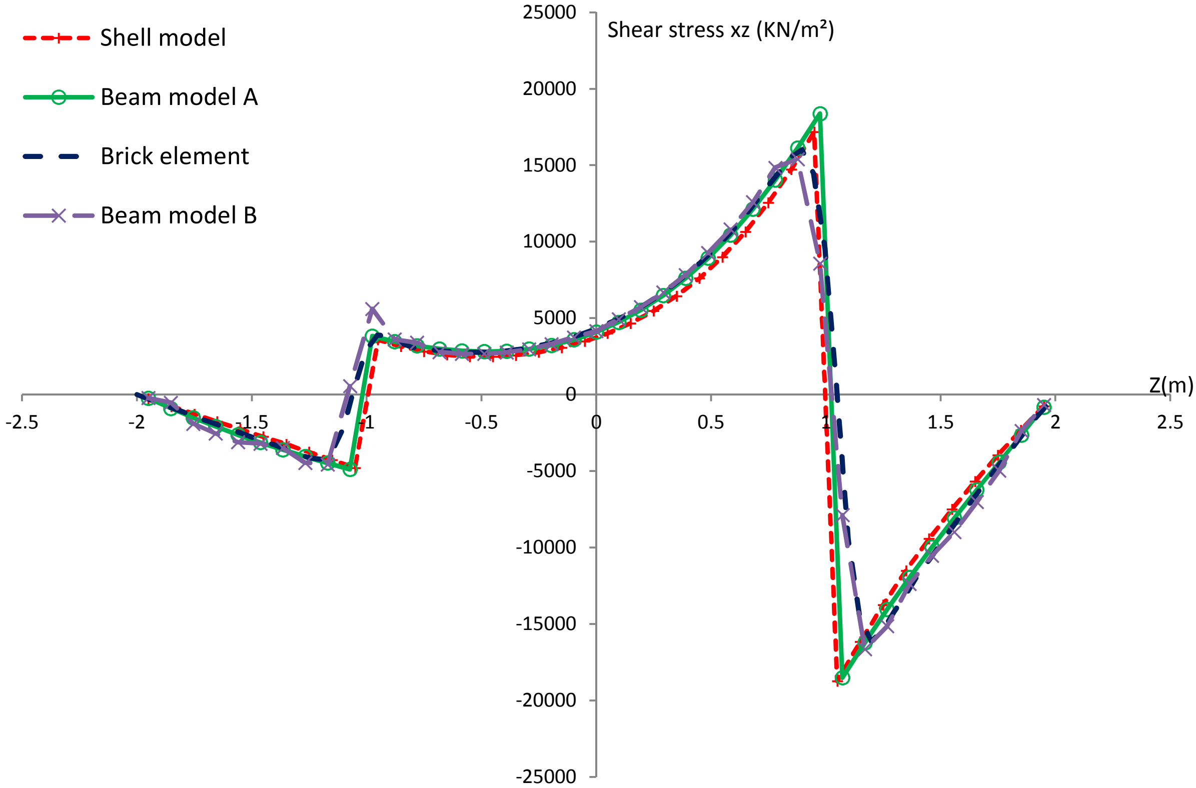


Figure 20 : Comparison of the shear stress xz between the brick shell and beam models, at x= 0.95m and at mid-depth of the upper slab (Ty = -10 MN)

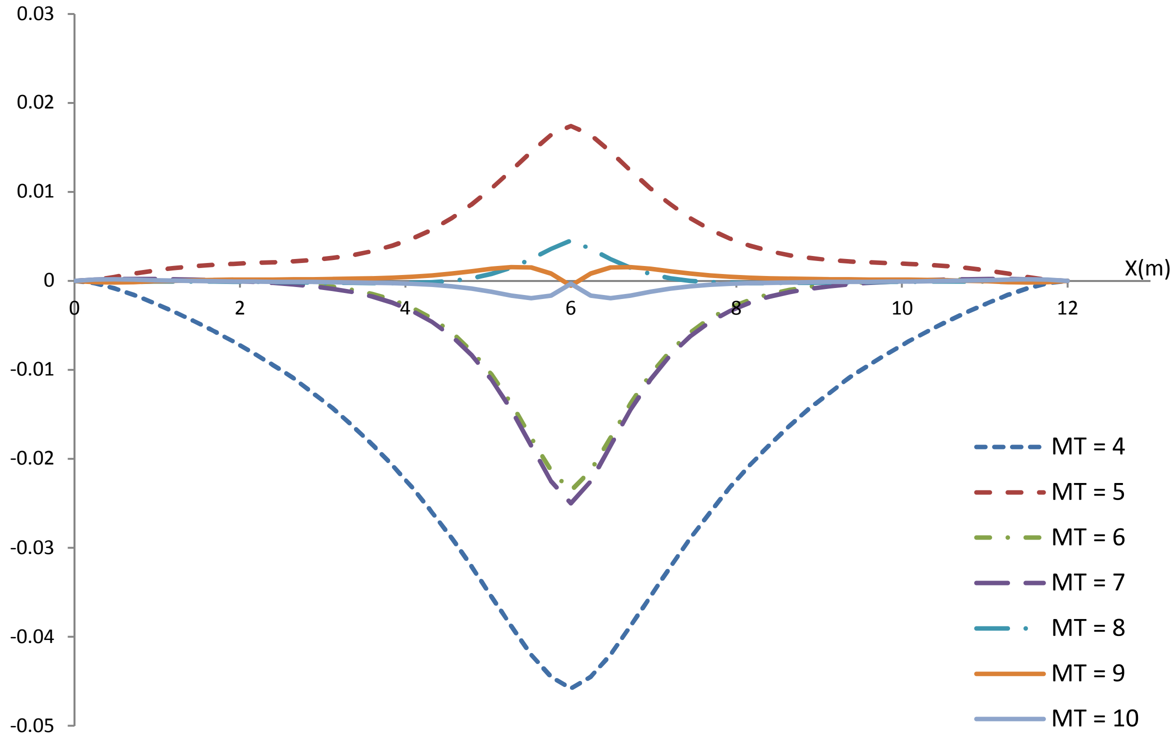


Figure 21 : representation of the generalized coordinates associated to the higher transversal modes(4-10) along the beam's length, for the beam model C in Fig.18.