A HIGHER ORDER BEAM FINITE ELEMENT WITH WARPING EIGENMODES

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Abstract:

In this paper, a new beam finite element is presented, with an accurate representation of normal stresses caused by "shear lag" or restrained torsion. This is achieved using an enriched kinematics, representing cross-section warping as the superposition of "warping modes". Detailed definitions and computational methods are given for these associated "warping functions". The exact solution of the equilibrium equations is given for a user-defined number of warping modes, though elastic results are totally mesh-independent.

keywords : shear lag, restrained torsion, warping, finite element method, beam.

1. Introduction:

In bridge engineering, it is generally needed to analyse the effect of torsional warping and shear lag on the stress distribution of beam cross-sections. This can not be achieved by using a model of classical beam finite element, based on either Bernoulli or Timoshenko theory. Two differents approaches are usual: The first is based on shell element models, that can be costly with respect to engineer time or computer time calculation, whereas the second relies on analytical methods, based for example on a Fourier series decomposition of forces (see Fauchart [8]), which is valid only for one-span system, can miss some effect when the section is not bi-symetric, and can hardly be integrated in finite element programs. The lack of an easy to use general method has motivated the present work to develop a new beam finite element able to describe very accurately the non-uniform warping of sections, either caused by non uniform torsion or shear lag.

The problem of warping have been widely treated in the existing litterature. In Bauchau[1], a similar approach of the one exposed here is used, consisting in ameliorating the Saint-Venant solution, that considers only the warping modes for a uniform warping, by adding new eigenwarping modes, derived from the principle of minimum potential energy. We propose here a different approach, that has the advantage to separate the determination of the warping modes from the equilibrium solution, and to propose a finite element formulation using this modes. Sapountzakis and Mokos[2] calculate a secondary shear stress, due to a non-uniform torsion warping, this can be considered here as the derivation of the second torsion warping mode, however in many cases this is not sufficient to represent accurately the stress distribution over the beam cross-section.

This paper presents a new kinematics for beams, that describe the out of plane displacements in the case of a non-uniform warping of a non-symetric section. This is achieved with using "warping functions" defined on the beam cross section. The warping functions are determined iteratively using equilibrium equations along the beam, leading to partials derivatives problems. This can be considered as a generalization of the work of Sapountzakis and Mokos(2003), where a secondary shear stress is considered, obtained by the equilibrium of the normal stress due to the non-uniform warping. In the present work this secondary shear stress would represent the second warping mode. The idea is therefore to go further, considering that this secondary shear stress will induce a new warping mode with its associated normal stresses, that can be at its turn equilibrated, inducing a third shear stress coresponding to the third warping mode… Iterative equilibrium scheme is continued until a sufficient number of mode is obtained to represent accurately the non-uniform warping effects.

In the second part of this work, the variational principle is used to determine the equilibrium equations, containing the new terms introduced by the warping. Analytical resolution of these equations will lead to results that are completely mesh-independent, and avoid shear locking problem in finite element formulation. The main difficulty to perform the exact solution of equilibrium equations is that the number of unknowns and thus of equations, depend on the number of the warping mode used. The size of the stiffness matrix will be then variable, equal to 12+2n, with n the number of warping modes.

Finally, the results are presented for different examples of beams, that will be compared to those obtained by a four noded shell elements(MITC-4) model of the beam.

2. Determination of the warping functions:

The beam is described on (x,y,z) axis system, x being the longitudinal axis, and y and z principle inertia axes, centered in the gravity center.
u_q, v_q, w_q are the displacements of a material point q along x,y,z axes.

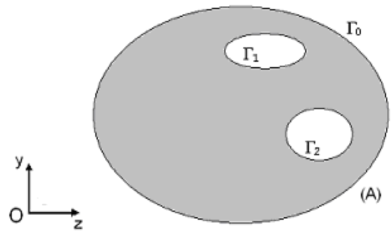


figure 1: cross section of the beam

Where A is the cross section area and $\Gamma = \bigcup_{0\le i} \Gamma_i$ the border of the section.

We assume the following displacement field of the beam :

$$u_q(x,y,z) = u(x) - y\theta_z(x) + z\theta_y(x) + \sum_{i=1}^{n} \Omega_i \xi_i(x) \quad (1)$$

$$v_q(x,y,z) = v(x) - z\theta_x(x) \quad (2)$$

$$w_q(x,y,z) = w(x) + y\theta_x(x) \quad (3)$$

Where Ω_i are the functions of the warping modes, and ξ_i the generalized coordinate associated to each mode.

Thus the resulting stress field for an homogenous cross section :

$$\sigma = E\left(\frac{du}{dx} - y\frac{d\theta_z}{dx} + z\frac{d\theta_y}{dx} + \sum_{i=1}^{n} \Omega_i \frac{d\xi_i}{dx}\right) \quad (4)$$

$$\tau_{xy} = G\left(\frac{dv}{dx} - \theta_z - z\frac{d\theta_x}{dx} + \sum_{i=1}^{n} \frac{\partial \Omega_i}{\partial y}\xi_i\right) \quad (5)$$

$$\tau_{xz} = G\left(\frac{dw}{dx} + \theta_y + y\frac{d\theta_x}{dx} + \sum_{i=1}^{n} \frac{\partial \Omega_i}{\partial z}\xi_i\right) \quad (6)$$

Where E and G are respectively the elasticity and shear modulus.

The following sections will present in details the derivation of the different warping modes, characterized by their Ω functions.

2.1. 1st modes determination:

For the derivation of 1st warping modes, it is necessary to distinguish between those due to shear, and the one related to torsion.
Let's start with the 1st warping mode for a shear force along y.
In the case where the beam is submitted to a uniform warping along the beam, due only to a bending in the xy plane, ξ will be constant and we take it equal to 1.
The displacement field becomes :

$$u_q = -y\theta_z(x) + \Omega_{1y} \tag{7}$$

$$v_q = v(x) \tag{8}$$

$$w_q = 0 \tag{9}$$

Thus the resulting stress field :

$$\sigma = -E\frac{d\theta_z}{dx}y \tag{10}$$

$$\tau_{xy} = G\left(\frac{dv}{dx} - \theta_z + \frac{\partial\Omega_{1y}}{\partial y}\right) \tag{11}$$

$$\tau_{xz} = G\frac{\partial\Omega_{1y}}{\partial z} \tag{12}$$

Assuming no body forces, the equilibrium equation is written as :

$$\frac{\partial\sigma}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0 \tag{13}$$

Substituting the stresses with their expressions, it comes :

$$\Delta\Omega_{1y} = \frac{T_y}{GI_z}y \tag{14}$$

With $\Delta = \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator, I_z the moment of inertia and T_y the shear force along y.

The warping must not generate either normal force or bending moment, which leads to the following orthogonalization equations :

$$\int_A \Omega_{1y}\,dA = 0 \tag{15}$$

$$\int_A y\,\Omega_{1y}\,dA = 0 \tag{16}$$

$$\int_A z\,\Omega_{1y}\,dA = 0 \tag{17}$$

With these additional conditions, the solution of (14) will be unique.

Using the same method to derive the 1st warping mode for a shear effort along z, we will have to resolve the following partial derivative problem:

$$\Delta\Omega_{1z} = \frac{T_z}{GI_y}z \tag{18}$$

$$\int_A \Omega_{1z} dA = 0 \quad (19)$$

$$\int_A z\,\Omega_{1z} dA = 0 \quad (20)$$

$$\int_A y\,\Omega_{1z} dA = 0 \quad (21)$$

The detail for the derivation of the 1st torsion mode of the Vlassov theory is given in [5] and in [4,6,7] for thin walled section. We give here the stated problem:

$$\Delta\Omega_{1t} = 0 \qquad \text{on } A \quad (22)$$

$$\frac{\partial\Omega_{1t}}{\partial n} = y n_z - z n_y \qquad \text{on } \Gamma \quad (23)$$

$$\int_A \Omega_{1t} dA = 0 \quad (24)$$

All of this partial derivatives problems can be resolved by different methods as finite difference(FDM), finite element(FEM) or boundary element method(BEM).

2.2. Determination of the warping function for some modes:

In the case of a non-uniform warping, the 1st modes will not be enough to describe the warping of a section, especially in the vicinity of a support section where warping is constrained. This is because the 1st warping modes are calculated by equilibrating the normal stress($\sigma = \frac{M_z}{I_z} y - \frac{M_y}{I_y} z$ for bending) for a uniform warping(ξ=cst), but in a non uniform case we will have $\frac{d\xi}{dx} \neq 0$, which will lead to the apparition of a warping normal stress $\sigma = E\frac{d\xi}{dx}\Omega$ that are not equilibrated in eq. (13).

Restoring equilibrium leads to the determination of a secondary shear stress associated to a 2nd warping mode. This reasoning can be considered as an iterative equilibrium schemes, converging to the exact shape of the warping in a section.

We assume that we have determined the nth warping mode, whether for shear or torsion, and we wish to determine the n+1th warping mode. The nth warping normal stress σ^n will be equilibrated by the n+1th warping shear stress :

$$\frac{\partial\sigma^n}{\partial x} + \frac{\partial\tau_{xy}^{n+1}}{\partial y} + \frac{\partial\tau_{xz}^{n+1}}{\partial z} = 0 \quad (25)$$

Where : $\sigma^n = E\frac{d\xi_n}{dx}\Omega_n \quad ; \quad \tau_{xy}^{n+1} = G\xi_{n+1}\frac{\partial\Omega_{n+1}}{\partial y} \quad ; \quad \tau_{xz}^{n+1} = G\xi_{n+1}\frac{\partial\Omega_{n+1}}{\partial z}$

Thus :

$$E\Omega_n \frac{d^2\xi_n}{dx^2} + G\xi_{n+1}\Delta\Omega_{n+1} = 0 \quad (26)$$

The functions Ω_{n+1} and Ω_n depends only of the geometry of the cross section, whereas ξ_{n+1} and ξ_n depends of the abscissa x, so eq. 26 implies that it necessarily exists two constants γ_{n+1} and β_{n+1}, related to the equilibrium of the beam, verifying: $\Delta\Omega_{n+1} = \gamma_{n+1}\Omega_n \quad ; \quad \xi_{n+1} = \beta_{n+1}\frac{d^2\xi_n}{dx^2}$.

Our goal is to construct a base of warping functions, where any section warping can be decomposed linearly with the aid of the ξ_i coefficients, that can be seen as the participation rate of the warping modes. In practice we need only to determine the warping functions to a multiplicative constant, and the participation rate for each mode will be obtained by writing the equilibrium of the beam. Thus, only the problem $\Delta\Omega_{n+1}=\Omega_n$ has to be solved. More details on the resolution of this partial deivative problem are given in appendix B.

Ω_{n+1} has to comply with the orthogonality conditions with respect to the n warping functions of the lower modes, this will assure the uniqueness of the function. To this aim the Gram-Scmidt orthogonalization process can be used:

$$\Omega_{n+1}^{j+1} = \Omega_{n+1}^{j+1} - \frac{\int_A \Omega_{n+1}^{j}\,\Omega_j \, dA}{\int_A \Omega_j^2 \, dA}\,\Omega_j \qquad for\ j = 1, n$$

At the end of the process we have Ω_{n+1}^{n+1}, which is the researched n+1[th] orthogonalized warping function.

3. Equilibrium equations and their resolutions:

3.1. Determination of the equilibrium equations:

The internal virtual work may be written as:

$$\delta W_{\text{int}} = \int_V \sigma^T \,\delta\varepsilon \, dV \tag{27}$$

Where V is beam's volume, $\sigma^T = (\sigma_x \quad \tau_{xy} \quad \tau_{xz})$ the stress vector and $\delta\varepsilon^T = (\delta\varepsilon_x \quad \delta\varepsilon_{xy} \quad \delta\varepsilon_{xz})$ the virtual strain vector.

Using the expression of the strain in the internal virtual work:

$$\begin{aligned}\delta W_{\text{int}} = &\int_V \left(\sigma_x \frac{d\delta u}{dx} - y\sigma_x \frac{d\delta\theta_z}{dx} + z\sigma_x \frac{d\delta\theta_y}{dx} + \sum_{i=1}^{n} \Omega_i \sigma_x \frac{d\delta\xi_i}{dx} \right) dV + \\ &\int_V \left(\tau_{xy}\left(\frac{d\delta v}{dx} - \delta\theta_z \right) + \tau_{xz}\left(\frac{d\delta w}{dx} + \delta\theta_y \right) + \frac{d\delta\theta_x}{dx}\left(y\tau_{xz} - z\tau_{xy} \right) + \sum_{i=1}^{n} \delta\xi_i \left(\tau_{xy}\frac{\partial\Omega_i}{\partial y} + \tau_{xz}\frac{\partial\Omega_i}{\partial z} \right) \right) dV\end{aligned} \tag{28}$$

After integrating over the whole section, it comes:

$$\begin{aligned}\delta W_{\text{int}} = &\int_L \left(N\frac{d\delta u}{dx} + M_z \frac{d\delta\theta_z}{dx} + M_y \frac{d\delta\theta_y}{dx} + \sum_{i=1}^{n} B_i \frac{d\delta\xi_i}{dx} \right) dx + \\ &\int_L \left(T_y\left(\frac{d\delta v}{dx} - \delta\theta_z \right) + T_z\left(\frac{d\delta w}{dx} + \delta\theta_y \right) + M_x \frac{d\delta\theta_x}{dx} + \sum_{i=1}^{n} \varphi_i \delta\xi_i \right) dx\end{aligned} \tag{29}$$

Where L is the beam's length.

The expressions of the generalized stresses are:

$$N = \int_A \sigma_x dA \tag{30}$$

$$M_z = \int_A -y\sigma_x dA = EI_z \frac{d\theta_z}{dx} \tag{31}$$

$$M_y = \int_A z\sigma_x dA = EI_y \frac{d\theta_y}{dx} \tag{32}$$

$$B_i = \int_A \Omega_i \sigma_x dA = E\sum_{j=1}^{n} K_{i,j} \frac{d\xi_j}{dx} \tag{33}$$

$$T_y = \int_A \tau_{xy} dA = G\left(\left(\frac{dv}{dx} - \theta_z\right)A + \sum_{i=1}^{n} \xi_i P_i\right) \tag{34}$$

$$T_z = \int_A \tau_{xz} dA = G\left(\left(\frac{dw}{dx} + \theta_y\right)A + \sum_{i=1}^{n} \xi_i Q_i\right) \tag{35}$$

$$M_x = \int_A \left(y\tau_{xz} - z\tau_{xy}\right) dA = G\left(I_0 \frac{d\theta_x}{dx} + \sum_{i=1}^{n} \xi_i N_i\right) \tag{36}$$

$$\varphi_i = \int_A \left(\tau_{xy}\frac{\partial \Omega_i}{\partial y} + \tau_{xz}\frac{\partial \Omega_i}{\partial z}\right) dA = G\left(\left(\frac{dv}{dx} - \theta_z\right)P_i + \left(\frac{dw}{dx} + \theta_y\right)Q_i + \frac{d\theta_x}{dx} N_i + \sum_{j=1}^{n} \xi_i M_{j,i}\right) \tag{37}$$

Where $I_0 = I_y + I_z$ is the polar inertia, and the warping-related coefficients are :

$$K_{i,j} = \int_A \Omega_i \Omega_j dA \;\; ; \;\; P_i = \int_A \frac{\partial \Omega_i}{\partial y} dA \;\; ; \;\; Q_i = \int_A \frac{\partial \Omega_i}{\partial z} dA \;\; ; \;\; N_i = \int_A \left(y\frac{\partial \Omega_i}{\partial z} - z\frac{\partial \Omega_i}{\partial y}\right) dA \;\; ; \;\; M_{i,j} = \int_A \nabla\Omega_i \cdot \nabla\Omega_j dA$$

The efforts due to warping are B_i and φ_i, respectively the bi-moment and the bi-shear, associated to the i^th^ warping mode.

After an integration by parts of the internal virtual work in equation(29), one obtains:

$$\delta W_{\text{int}} = \int_L \left(\frac{dN}{dx}\delta u + \left(-\frac{dM_z}{dx} - T_y\right)\delta\theta_z + \left(-\frac{dM_y}{dx} + T_z\right)\delta\theta_y + \sum_{i=1}^{n}\delta\xi_i\left(-\frac{dB_i}{dx} + \varphi_i\right) + \frac{dT_y}{dx}\delta v - \frac{dT_z}{dx}\delta w - \frac{dM_x}{dx}\delta\theta_x\right) dx$$
$$\underbrace{\left[N\delta u + T_y\delta v + T_z\delta w + M_x\delta\theta_x + M_y\delta\theta_y + M_z\delta\theta_z + \sum_{i=1}^{n} B_i\delta\xi_i\right]_0^L}_{\delta W_{ext}} \tag{38}$$

From the principal of virtual work $\delta W_{\text{int}} - \delta W_{ext} = 0$, it comes:

$$\int_L \left(\frac{dN}{dx}\delta u + \left(-\frac{dM_z}{dx} - T_y\right)\delta\theta_z + \left(-\frac{dM_y}{dx} + T_z\right)\delta\theta_y + \sum_{i=1}^{n}\delta\xi_i\left(-\frac{dB_i}{dx} + \varphi_i\right) + \left(\frac{dT_y}{dx}\right)\delta v - \left(\frac{dT_z}{dx}\right)\delta w - \left(\frac{dM_x}{dx}\right)\delta\theta_x\right) dx = 0 \tag{39}$$

This relation is valid for any admissible virtual displacements, then all the expressions between brackets have to be zero:

$$\frac{dM_z}{dx} + T_y = 0 \;\; ; \;\; \frac{dM_y}{dx} - T_z = 0 \;\; ; \;\; \frac{dN}{dx} = 0 \;\; ; \;\; \frac{dT_y}{dx} = 0 \;\; ; \;\; \frac{dT_z}{dx} = 0 \;\; ; \;\; \frac{dM_x}{dx} = 0 \tag{40}$$

$$\frac{dB_i}{dx} - \varphi_i = 0 \quad 1 \le i \le n \tag{41}$$

3.2. Eigenmodes of warping:

From the expressions of the shear efforts and the torsion moment, we have:

$$\frac{dv}{dx}-\theta_z = \frac{T_y}{AG}-\sum_{i=1}^{n}\xi_i\frac{P_i}{A} \tag{42}$$

$$\frac{dw}{dx}+\theta_y = \frac{T_z}{AG}-\sum_{i=1}^{n}\xi_i\frac{Q_i}{A} \tag{43}$$

$$\frac{d\theta_x}{dx} = \frac{M_x}{GI_0}-\sum_{i=1}^{n}\xi_i\frac{N_i}{I_0} \tag{44}$$

After substituting in the expression of the bi-shear φ_i in the equation (37), we obtain:

$$\varphi_i = \frac{P_i}{A}T_y+\frac{Q_i}{A}T_z+\frac{N_i}{I_0}M_x+G\sum_{j=1}^{n}\xi_j\left(M_{j,i}-\frac{N_iN_j}{I_0}-\frac{Q_iQ_j+P_iP_j}{A}\right) \tag{45}$$

The n equilibrium equations for warping efforts in (41), can now be re-written in a system of differential equations :

$$\begin{Bmatrix}\xi_1''\\ \vdots\\ \xi_n''\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}K_{1,1} & \cdots & K_{1,n}\\ & \ddots & \vdots\\ sym & & K_{n,n}\end{bmatrix}^{-1}\begin{bmatrix}\frac{P_1}{A} & \frac{Q_1}{A} & \frac{N_1}{I_0}\\ \vdots & \vdots & \vdots\\ \frac{P_n}{A} & \frac{Q_n}{A} & \frac{N_n}{I_0}\end{bmatrix}\begin{Bmatrix}T_y\\ T_z\\ M_x\end{Bmatrix}+\frac{G}{E}\begin{bmatrix}K_{1,1} & \cdots & K_{1,n}\\ & \ddots & \vdots\\ sym & & K_{n,n}\end{bmatrix}^{-1}\begin{bmatrix}M_{1,1}-\frac{N_1^2}{I_0}-\frac{Q_1^2+P_1^2}{A} & \cdots & M_{1,n}-\frac{N_1N_n}{I_0}-\frac{Q_1Q_n+P_1P_n}{A}\\ & \ddots & \vdots\\ sym & & M_{n,n}-\frac{N_n^2}{I_0}-\frac{Q_n^2+P_n^2}{A}\end{bmatrix}\begin{Bmatrix}\xi_1\\ \vdots\\ \xi_n\end{Bmatrix}$$

For what follows, we introduce some notations :

$$\{\xi\}=\begin{Bmatrix}\xi_1\\ \vdots\\ \xi_n\end{Bmatrix}\quad ;\quad [K]=\begin{bmatrix}K_{1,1} & \cdots & K_{1,n}\\ & \ddots & \vdots\\ sym. & & K_{n,n}\end{bmatrix}\quad ;\quad [P]=\frac{1}{E}[K]^{-1}\begin{bmatrix}\frac{P_1}{A} & \frac{Q_1}{A} & \frac{N_1}{I_0}\\ \vdots & \vdots & \vdots\\ \frac{P_n}{A} & \frac{Q_n}{A} & \frac{N_n}{I_0}\end{bmatrix}\quad ;\quad \{f\}=\begin{Bmatrix}T_y\\ T_z\\ M_x\end{Bmatrix}$$

$$[M]=\frac{G}{E}[K]^{-1}\begin{bmatrix}M_{1,1}-\frac{N_1^2}{I_0}-\frac{Q_1^2+P_1^2}{A} & \cdots & M_{1,n}-\frac{N_1N_n}{I_0}-\frac{Q_1Q_n+P_1P_n}{A}\\ & \ddots & \vdots\\ sym. & & M_{n,n}-\frac{N_n^2}{I_0}-\frac{Q_n^2+P_n^2}{A}\end{bmatrix}$$

We note that $[K]$ is the Gramian matrix attached to the warping functions, and since all diagonal terms are strictly positive, the matrix will be then positive definite and inversible whatever number of modes considered.

The system of differential equations can now be written in a compact matrix form :

$$\{\xi''\}=[P]\{f\}+[M]\{\xi\} \tag{46}$$

Let : $(\lambda_i)_{1\le i\le n}$ represent the eigenvalues of $[M]$, and $[R]$ the corresponding eingenvectors matrix.
Writing $[\xi]=[R]\{X\}$, we have:

$$[R]\{X''\}=[P]\{f\}+[M][R]\{X\}$$

$$\{X''\}=[R]^{-1}[P]\{f\}+[R]^{-1}[M][R]\{X\}$$
$$\{X''\}=[R]^{-1}[P]\{f\}+[\lambda]\{X\} \quad (47)$$

Where $[\lambda]$ is the diagonal matrix containing the eigenvalues.
The system being now uncoupled, it can be solved as :

$$\{X\}=\{z\}-[\lambda]^{-1}[R]^{-1}[P]\{f\} \quad (48)$$

Where $z_i = A_i ch(\sqrt{\lambda_i}x) + B_i sh(\sqrt{\lambda_i}x)$ and A_i , B_i integration constants depending on the boundary conditions.
Hence, $\{\xi\}$ is obtained as :

$$\{\xi\}=[R]\{z\}-[R][\lambda]^{-1}[R]^{-1}[P]\{f\} \Rightarrow \{\xi\}=[R]\{z\}-\underbrace{[M]^{-1}[P]}_{[H]}\{f\} \quad (49)$$

We re-write the solution of the system under the following matrix form :

$$\{\xi\}_x=[R]\left([ch]_x\{A\}+[sh]_x\{B\}\right)-[H]\{f\} \quad (50)$$

Where $[ch]_x=\begin{bmatrix} ch(\sqrt{\lambda_1}x) & & 0 \\ & \ddots & \\ 0 & & ch(\sqrt{\lambda_n}x)\end{bmatrix}$; $[sh]_x=\begin{bmatrix} sh(\sqrt{\lambda_1}x) & & 0 \\ & \ddots & \\ 0 & & sh(\sqrt{\lambda_n}x)\end{bmatrix}$; $\{A\}=\begin{Bmatrix} A_1 \\ \vdots \\ A_n\end{Bmatrix}$; $\{B\}=\begin{Bmatrix} B_1 \\ \vdots \\ B_n\end{Bmatrix}$

We will now determine the vectors {A} and {B} in function of the boundary conditions of {ξ} :

$$\{\xi\}_0=\{\xi_A\} \Rightarrow \{\xi_A\}=[R]\{A\}-[H]\{f\} \quad (51)$$
$$\{\xi\}_L=\{\xi_B\} \Rightarrow \{\xi_B\}=[R]\left([ch]_L\{A\}+[sh]_L\{B\}\right)-[H]\{f\} \quad (52)$$

Thus :

$$\{A\}=[R]^{-1}(\{\xi_A\}+[H]\{f\}) \quad (53)$$
$$\{B\}=[sh]_L^{-1}[R]^{-1}(\{\xi_B\}+[H]\{f\})-[th]_L^{-1}[R]^{-1}(\{\xi_A\}+[H]\{f\}) \quad (54)$$

If we note the hyperbolic matrices:

$$[H_1]_x=[R]\left([ch]_x-[sh]_x[th]_L^{-1}\right)[R]^{-1} \quad (55)$$
$$[H_2]_x=[R][sh]_x[sh]_L^{-1}[R]^{-1} \quad (56)$$

We can write $\{\xi\}$ as a function of its end values and the abscissa x in the following form:

$$\{\xi\}_x=[H_1]_x\{\xi_A\}+[H_2]_x\{\xi_B\}+\left([H_1]_x+[H_2]_x-I_n\right)[H]\{f\} \quad (57)$$

3.3. Resolution of the equilibrium equations and determination of the stiffness matrix:

In classical beam finite element formulation "arbitrary" interpolation functions are used for the displacements, and then variational principle is used to derive the stiffness matrix. The accuracy of the calculation results obtained with this formulation would be mesh dependent, especially for warping coordinates, which are of hyperbolic form, and we can also have shear locking problem for thin walled

beams, due to the fact that we can't assure exactly the constraints of zero shear deformations in every position in the beam.
In the following work a different approach is used to determine the stiffness matrix. From the resolution of the equilibrium equations, we will express the n+6 external generalized forces at each node of the beam, as a function of the 2n+12 nodal displacements. With these expressions the stiffness matrix can be assembled. Nevertheless, performing the exact solution has a major difficulty, consisting in that our stiffness matrix has a variable length, depending on the number of warping modes, but must be always derived from the exact solution of the equilibrium equations.
We write the 12+2n equations from the equilibrium equations (40) and (41) :

$$\frac{dT_y}{dx}=0 \Rightarrow T_y(x)=T_{yA} \qquad \text{in } x = L: \qquad T_{yA}=T_{yB} \quad (58) \qquad v(L)=v_B \quad (59)$$

$$\frac{dT_z}{dx}=0 \Rightarrow T_z(x)=T_{zA} \qquad \text{in } x = L: \qquad T_{zA}=T_{zB} \quad (60) \qquad w(L)=w_B \quad (61)$$

$$\frac{dM_x}{dx}=0 \Rightarrow M_x(x)=M_{xA} \qquad \text{in } x = L: \qquad M_{xB}=M_{xA} \quad (62) \qquad \theta_x(L)=\theta_{xB} \quad (63)$$

$$\frac{dM_z}{dx}+T_y=0 \Rightarrow M_z(x)=M_{zA}-T_{yA}.x \qquad \text{in } x = L: \qquad M_{zB}=M_{zA}-T_{yA}L \quad (64)$$

$$\theta_z(x)=\theta_{zA}+\frac{M_{zA}}{EI_z}x-\frac{T_{yA}}{2EI_z}x^2 \qquad \text{in } x = L: \qquad \theta_{zB}=\theta_{zA}+\frac{M_{zA}}{EI_z}L-\frac{T_{yA}}{2EI_z}L^2 \quad (65)$$

$$\frac{dM_y}{dx}-T_z=0 \Rightarrow M_y(x)=M_{yA}+T_{zA}x \qquad \text{in } x = L: \qquad M_{yB}=M_{yA}+T_{zA}L \quad (66)$$

$$\theta_y(x)=\theta_{yA}+\frac{M_{yA}}{EI_y}x+\frac{T_{zA}}{2EI_y}x^2 \qquad \text{in } x = L: \qquad \theta_{yB}=\theta_{yA}+\frac{M_{yA}}{EI_y}L+\frac{T_{zA}}{2EI_y}L^2 \quad (67)$$

$$\frac{dN}{dx}=0 \Rightarrow N(x)=N_A \qquad \text{in } x = L: \qquad N_A=N_B \quad (68) \qquad u(L)=u_B \quad (69)$$

And the additional 2n equations for the bi-moment :

$$B_i(0)=B_{iA} \quad ; \quad B_i(L)=B_{iB} \text{ for } 1\le i\le n \qquad (70)$$

We have then 2n +12 equations for 2n+12 unknowns.
The equations (59), (61), (63), (69) and those of biforce (70), need to be developped more explicitly.
For equation (59) :

$$\frac{dv}{dx}=\theta_z+\frac{T_y}{AG}-\sum_{i=1}^{n}\xi_i\frac{P_i}{A}$$

$$\frac{dv}{dx}=\theta_{zA}+\frac{M_{zA}}{EI_z}x-\frac{T_{yA}}{2EI_z}x^2+\frac{T_{yA}}{AG}-\frac{1}{A}\{P\}^T\{\xi\} \qquad (71)$$

Where $\{P\}^T=\{P_1 \ \cdots \ P_n\}$

Integrating (71) from 0 to L:

$$-\frac{M_{zA}}{2EI_z}L^2+T_{yA}\left(\frac{L^3}{6EI_z}-\frac{L}{GA}\right)+\frac{2}{A}\{P\}^T[T_1][H]\{f\}-\frac{L}{A}\{P\}^T[H]\{f\}=v_A-v_B+\theta_{zA}L+\frac{1}{A}\{P\}^T[T_1](\{\xi_A\}+\{\xi_B\}) \quad (72)$$

With : $[T_1]=\int_0^L[H_1]_x\,dx=\int_0^L[H_2]_x\,dx=[R]\left[\frac{1}{\sqrt{\lambda_i}}g\left(\sqrt{\lambda_i}L\right)\delta_{ij}\right]_{1\le i,j\le n}[R]^{-1}$ and $g(x)=\frac{1}{th(x)}-\frac{1}{sh(x)}$

The same method is applied for the equation (61) and (63), leading to the following equations :

$$\frac{M_{yA}}{2EI_y}L^2+T_{zA}\left(\frac{L^3}{6EI_y}-\frac{L}{GA}\right)+\frac{2}{A}\{Q\}^T[T_1][H]\{f\}-\frac{L}{A}\{Q\}^T[H]\{f\}=w_A-w_B-\theta_{yA}L+\frac{1}{A}\{Q\}^T[T_1](\{\xi_A\}+\{\xi_B\}) \quad (73)$$

$$-\frac{M_{xA}L}{GI_0}+\frac{2}{I_0}\{N\}^T[T_1][H]\{f\}-\frac{L}{I_0}\{N\}^T[H]\{f\}=\theta_{xA}-\theta_{xB}+\frac{1}{I_0}\{N\}^T[T_1](\{\xi_A\}+\{\xi_B\}) \quad (74)$$

Where $\{Q\}^T=\{Q_1 \ \cdots \ Q_n\}$; $\{N\}^T=\{N_1 \ \cdots \ N_n\}$.
For the 2n equations in (70), related to bi-moment :

$$\{B(0)\}=\{B_A\} \quad \Rightarrow \quad E[K]\{\xi'\}_0=\{B_A\} \quad (75)$$
$$\{B(L)\}=\{B_B\} \quad \Rightarrow \quad E[K]\{\xi'\}_L=\{B_B\} \quad (76)$$

From the expression of {ξ} it comes :

$$\{\xi'\}_0=[T_2]\{\xi_A\}+[T_3]\{\xi_B\}+([T_2]+[T_3])[H]\{f\} \quad (77)$$
$$\{\xi'\}_L=-[T_3]\{\xi_A\}-[T_2]\{\xi_B\}-([T_2]+[T_3])[H]\{f\} \quad (78)$$

Where: $[T_2]=\left(\frac{d}{dx}[H_1]_x\right)_{x=0}=-\left(\frac{d}{dx}[H_2]_x\right)_{x=L}=[R]\left[\sqrt{\lambda_i}\,h\left(\sqrt{\lambda_i}L\right)\delta_{ij}\right]_{1\le i,j\le n}[R]^{-1}$ and $h(x)=\frac{-1}{th(x)}$

$[T_3]=\left(\frac{d}{dx}[H_2]_x\right)_{x=0}=-\left(\frac{d}{dx}[H_1]_x\right)_{x=L}=[R]\left[\sqrt{\lambda_i}\,t\left(\sqrt{\lambda_i}L\right)\delta_{ij}\right]_{1\le i,j\le n}[R]^{-1}$ and $t(x)=\frac{1}{sh(x)}$

We note that we have : $[\lambda][T_1]+[T_2]+[T_3]=0$
Thus we write :

$$\{B_A\}+E[K][\lambda][T_1][H]\{f\}=[T_2]\{\xi_A\}+[T_3]\{\xi_B\} \quad (79)$$

$$\{B_B\}-E[K][\lambda][T_1][H]\{f\}=-[T_3]\{\xi_A\}-[T_2]\{\xi_B\} \quad (80)$$

And finally for equation (69):

$$\frac{du}{dx} = \frac{N_A}{EA} \quad \Rightarrow \quad \frac{N_A}{EA} L = u_B - u_A \tag{81}$$

By assembling all of the 2n+12 equations, we obtain the following system :

$$[K_F]\{\Psi\} = [K_D]\{d\} \tag{82}$$

Where {d} and {Ψ} represent, respectively, the displacements and the generalized efforts vector.
We deduce the stiffness matrix $[K_S]$:

$$[K_S] = [K_F]^{-1} [K_D] \tag{83}$$

4. Numerical examples:

Two examples are presented below, both being a 10m-length cantilever beam, Young's modulus E=40Gpa, Poisson's ration υ=0, but with different cross sections. The comparisons will be performed with a finite element model of the cantilever beam using MITC 4 noded shell elements, described in [10].

We only compare the normal stress due to warping :

- On the shell finite element model, warping normal stress is obtained by deducting an assumed linear stress state from the actual calculated stress.
- On the beam model, we use the stress calculated by $\sigma = E\sum \Omega_i \frac{d\xi_i}{dx}$.

The boundary conditions for this example are:

- Restrained warping at the beam's bearing: $\xi_i = 0$
- No warping restraining at the free end: $\frac{d\xi_i}{dx} = 0$

The comparison of the normal stresses between the shell and the beam model is carried out at x=0.05m from the fixed end, sufficiently far from load application point to respect the Saint-Venant principle, and where the restrained-warping effect is important. The higher eigenmodes of warping will not be neglectable and so we can see their effects on the normal stress.
For the following examples, we will mean by 'beam model(iY jZ kT)', a model with beam finite element with i-warping modes of shear along y, j-warping modes of shear along z and k-warping modes of torsion.

4.1. Box girder:

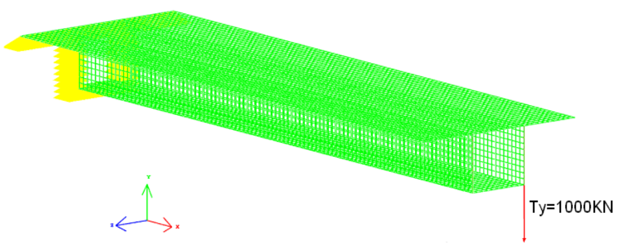


Figure 2 : Shell model of the beam with an external load Ty = -1000 kN

The comparaison of the normal stresses in the section will be carried out at mid-thickness of the upper slab.

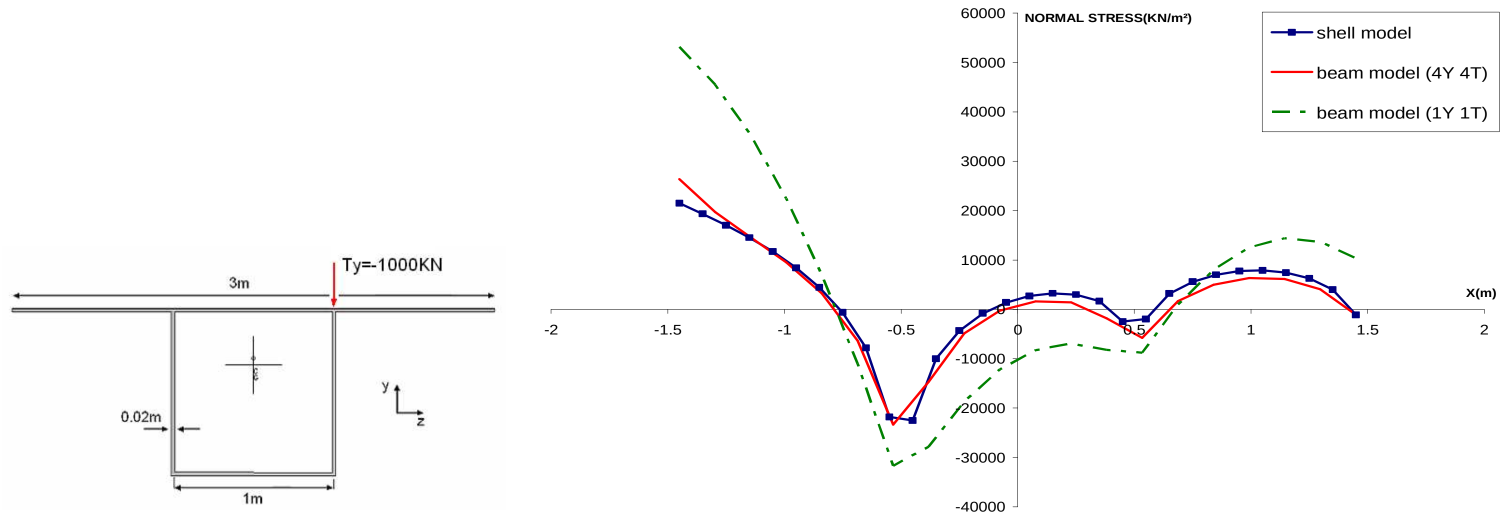


Figure 3 : Comparison of the normal stresses between the shell and the beam model, at x = 0.05m and at mid-depth of the upper slab(Ty = -1000 kN)

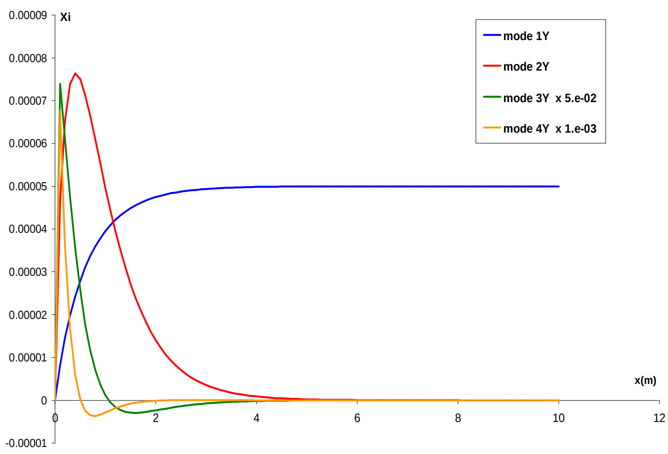


figure 4: shear along y warping d.o.f. along the beam

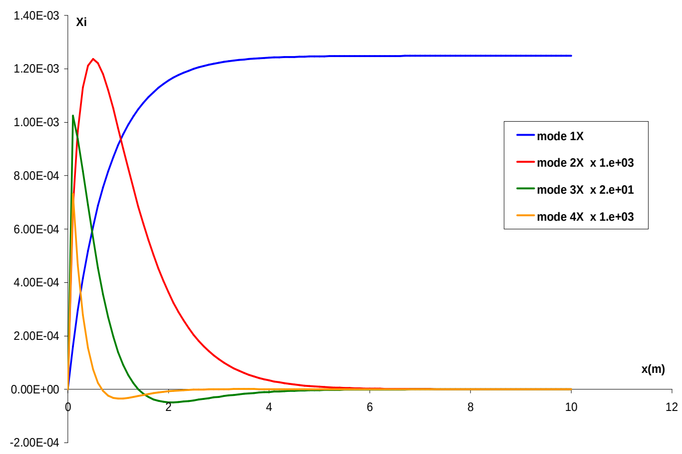


figure 5: torsion warping d.o.f. along the beam

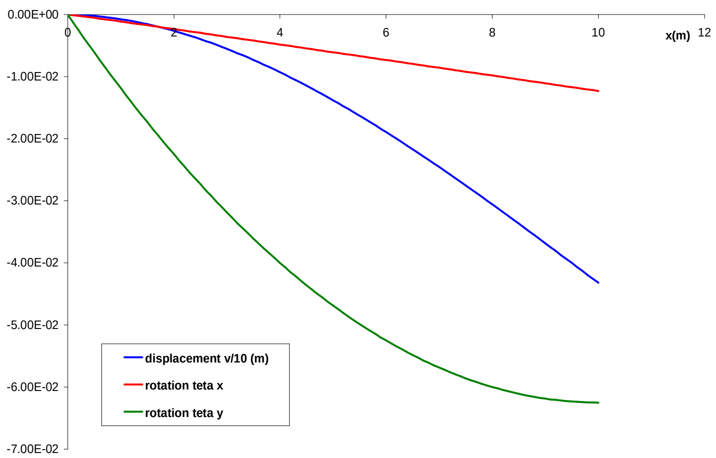
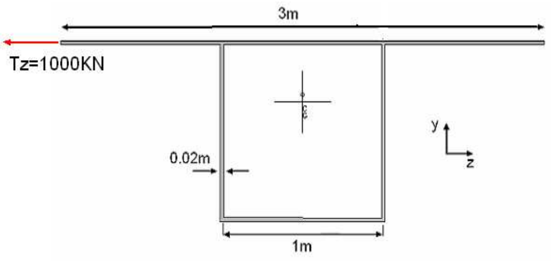


figure 6: displacement and rotations of the beam

From the figures 4 and 5, we can clearly see that the effect of the higher warping modes will be non negligeable in the fixed end, and disapear when we moves away from it. If we have used interpolation functions, it will have been necessary to use a refined meshing near the fixed end, to obtain the higher order mode with precision, this shows the advantage of using an exact solution of the equilibrium equations to construct the stiffness matrix.



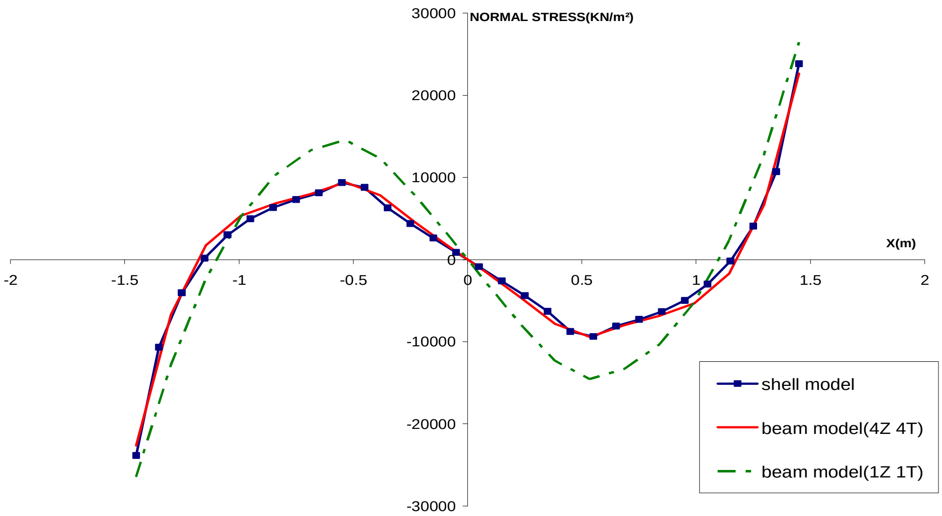


Figure 7 : Comparison of the normal stresses between the shell and the beam model, at x = 0.05m and at mid-depth of the upper slab (Tz = 1000 kN)

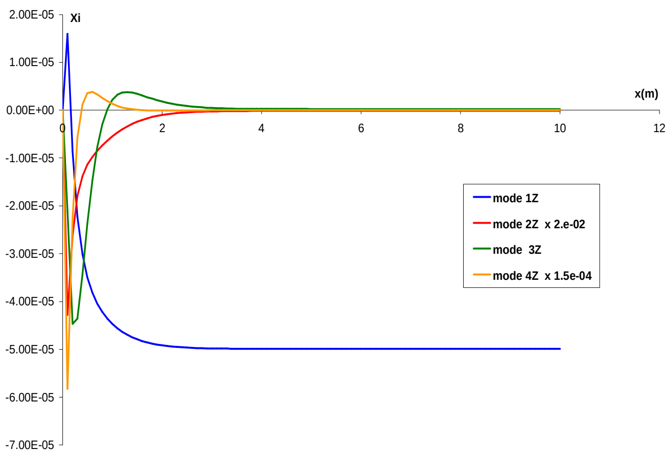


figure 8: shear along z warping d.o.f. along the beam

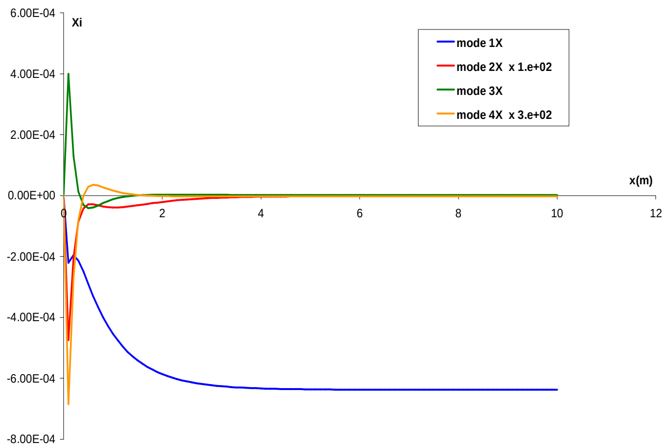


figure 9: torsion warping d.o.f. along the beam

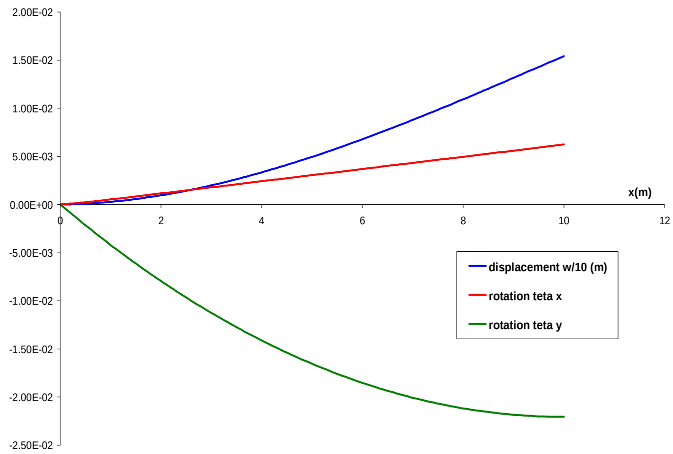


figure 10: displacement and rotations of the beam

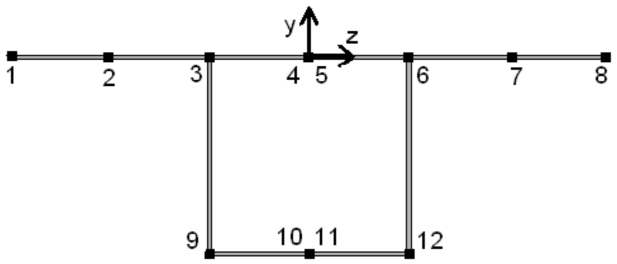


Figure 11: measure points in the cross section

table 1: coordinate of the measure points

	y(m)	z(m)
1	0	-1.45
2	0	-0.95
3	0	-0.45
4	0	-0.05
5	0	0.05
6	0	0.45
7	0	0.95
8	0	1.45
9	-1	-0.45
10	-1	-0.05
11	-1	0.05
12	-1	0.45

	Measure points											
	1	2	3	4	5	6	7	8	9	10	11	12
shell	21516	8431	-22528	1385	2705	-2452	7737	-1136	12777	-8483	-8361	13505
Beam 4Y4T	26372	7970	-22026	220	1437	-4940	6103	-1146	7152	-9463	-8571	18909
Beam 1Y1T	53175	18457	-32755	-11310	-8744	-9661	11540	10363	3144	-9404	-6977	24986

Table 2: normal stress(KN/m²) in the measure points for Ty=-1000KN

	Measure points											
	1	2	3	4	5	6	7	8	9	10	11	12
shell	-23844	4961	8774	869	-869	-8774	-4961	23844	-5339	-177	177	5339
Beam 4Z4T	-22633	5859	9360	958	-965	-9361	-5855	22640	-7579	-240	243	7581
Beam 1Y1T	-26451	6856	14069	1743	-1741	-14067	-6856	26449	-10055	-938	939	10056

Table 3: normal stress(KN/m²) in the measure points for Tz=1000KN

4.2. I-beam:

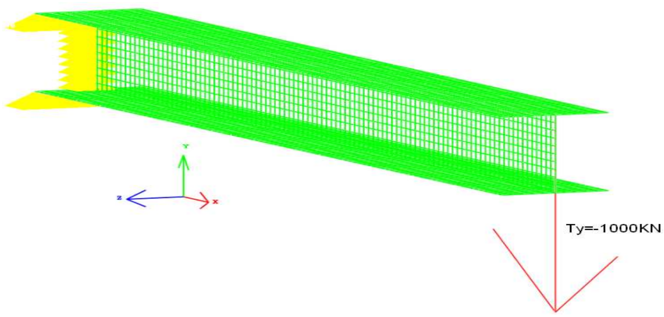
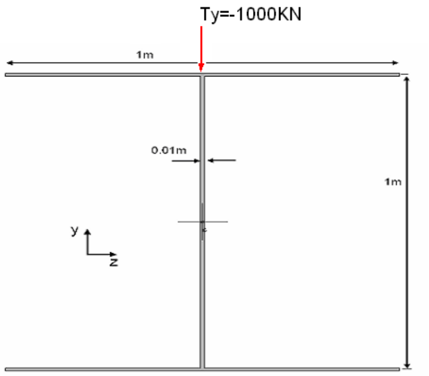


Figure 12 : shell model of the beam with an external load Ty = -1000 kN



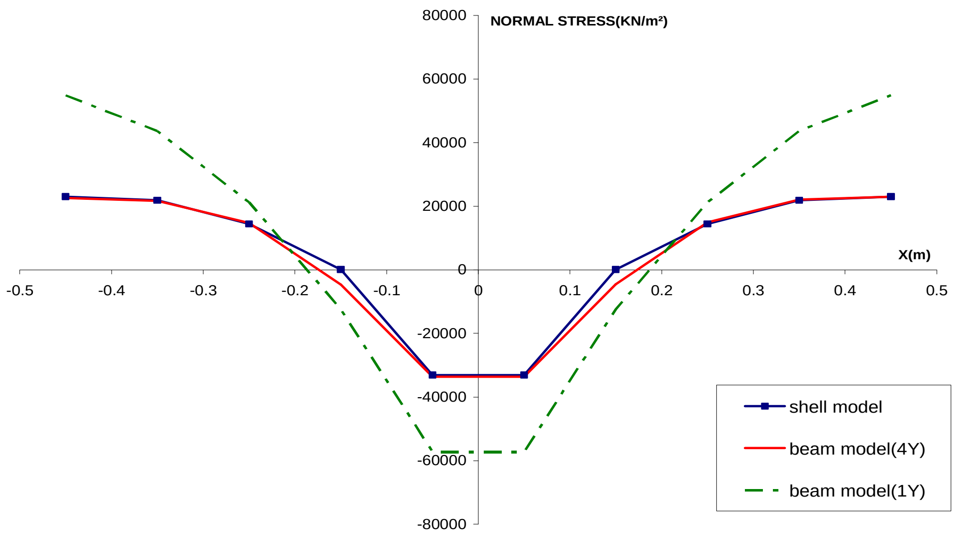


Figure 13 : Comparison of the normal stresses between the shell and the beam model, at x = 0.05m and at mid-depth of the upper slab (Ty = -1000 kN)

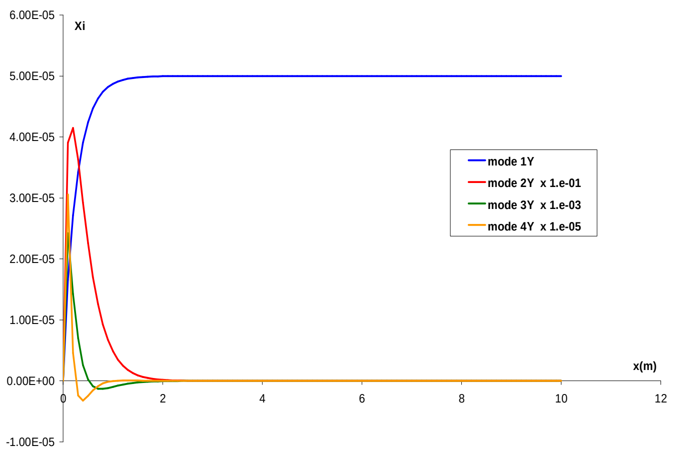


figure 14: shear along y warping d.o.f. along the beam

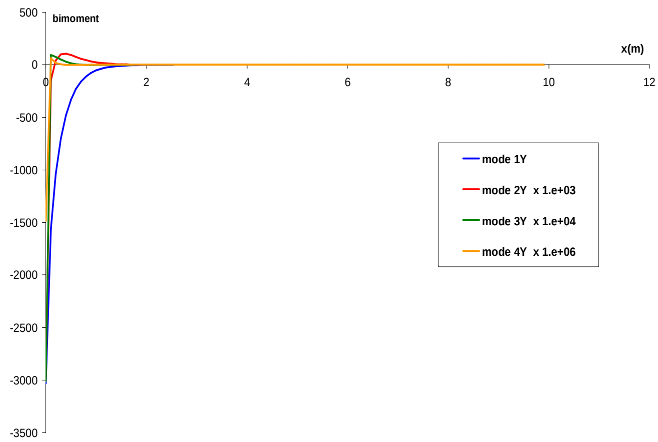


figure 15: bi-moments along the beam

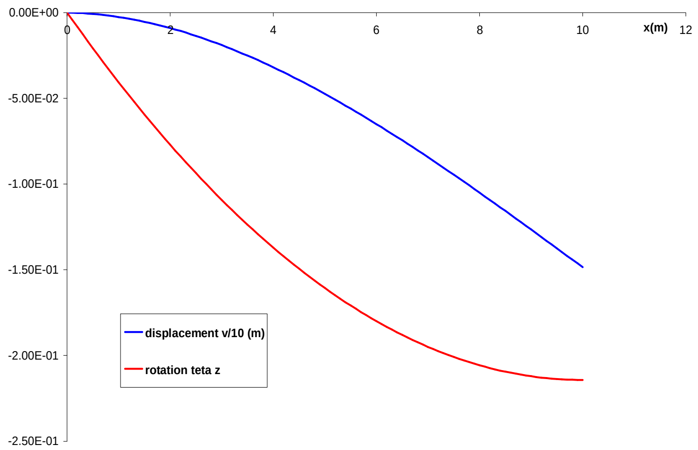


figure 16: displacement and rotation of the beam

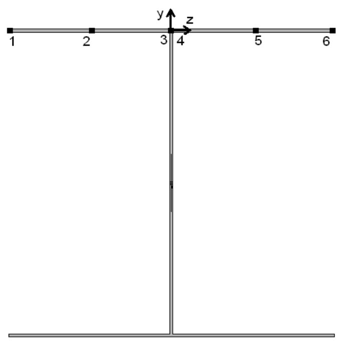


Figure 17: measure points in the cross section

table 4: coordinate of the measure points

	y(m)	z(m)
1	0	-0.45
2	0	-0.25
3	0	-0.05
4	0	0.05
5	0	0.25
6	0	0.45

	Measure points					
	1	2	3	4	5	6
shell	23022	14371	-33110	-33110	14371	23022
Beam 4Y4T	22630	14742	-33668	-33627	14986	22988
Beam 1Y1T	54827	21204	-57258	-57267	21206	54841

Table 5: normal stress(KN/m^2) in the measure points for Ty=-1000KN

5. Conclusion:

A new beam element has been derived, allowing an accurate representation of the restrained warping effect. It can be used for shear lag representation or restrained torsion. We note that all the stresses measures performed in the numerical exemples were done in the vicinity of the support section at x=L/200, the results shows that we can't neglect the effect of the higher warping modes, if we want to obtain an accurate description of warping.

The number of additional d.o.f. is user-determined. The element has shown very precise results with 4 warping parameters at each node, for torsion and for each shear direction – total 24 additional d.o.f. on the element.

Longitudinal interpolation is exact for linear-elastic behaviour, so that the results are totally mesh-independent. This important feature allows the use of this new element with coarse discretization, in a similar way as Euler-Bernoulli traditional elements.

The formulation used here can be generalized easily to anisotropic materials, the main difference will be in the derivation of the warping functions.

Appendix A:

We give here some examples of warping modes for different section.

Rectangular section :

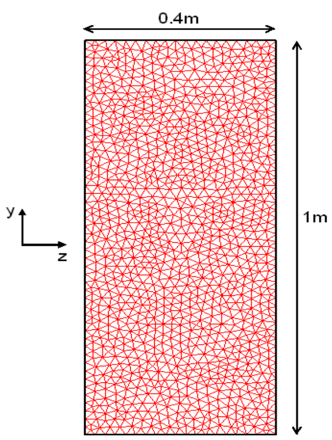


Figure 1 : Section mesh, 2164 triangular elements and 1151 nodes.

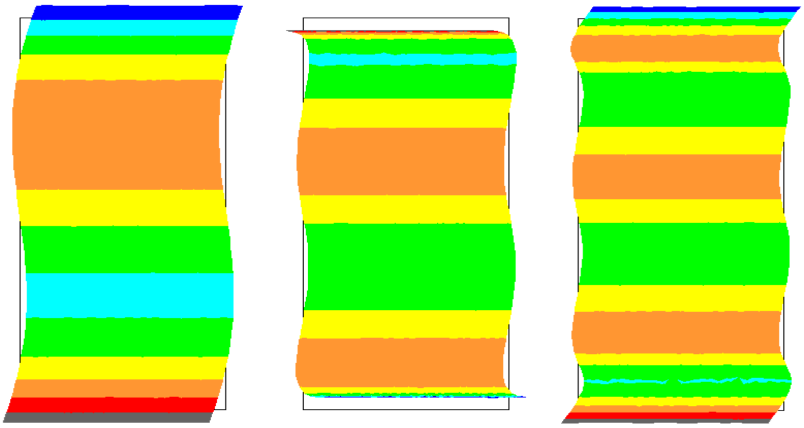


Figure 2 : 3 first warping modes of shear along y.

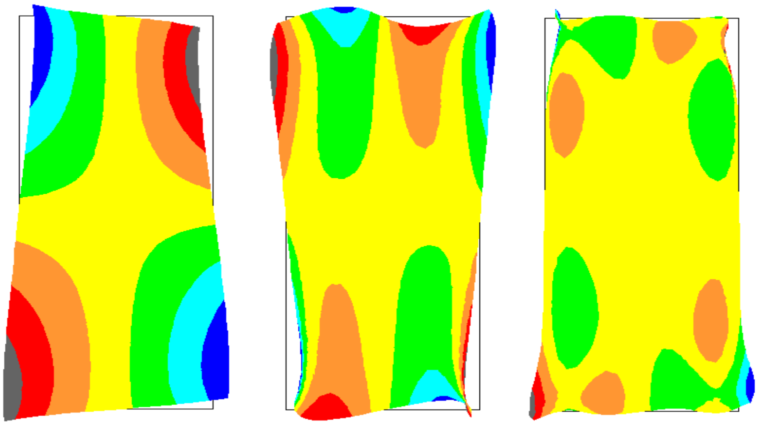


Figure 3 : 3 first warping modes of torsion.

I section :

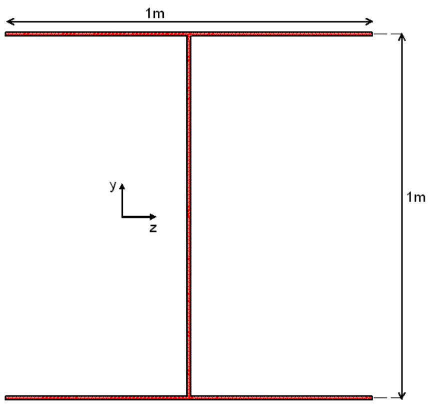


Figure 4 : Section mesh, 2384 triangular elements and 1788 nodes.

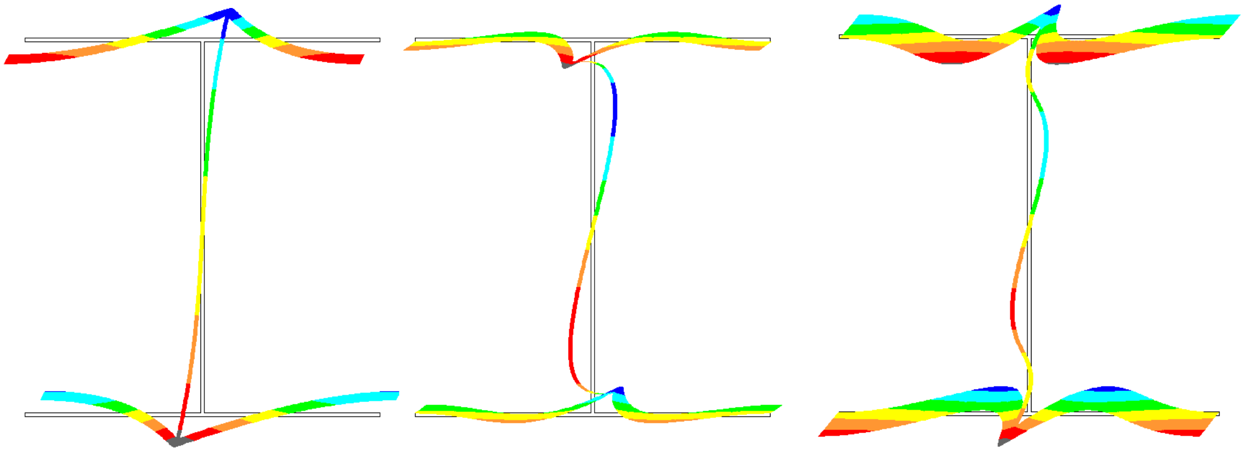


Figure 5 : 3 first warping modes of shear along y.

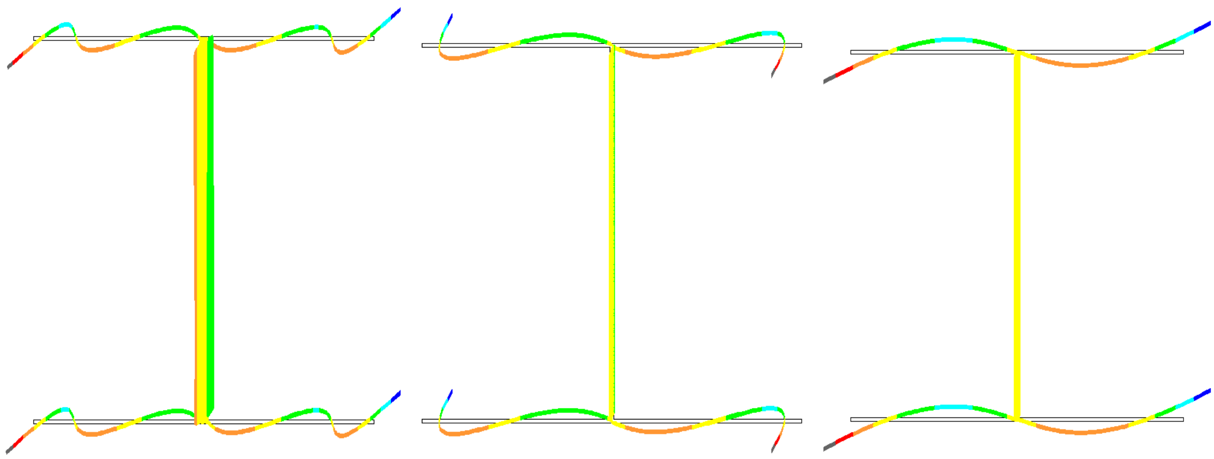


Figure 6 : 3 first warping modes of shear along z.

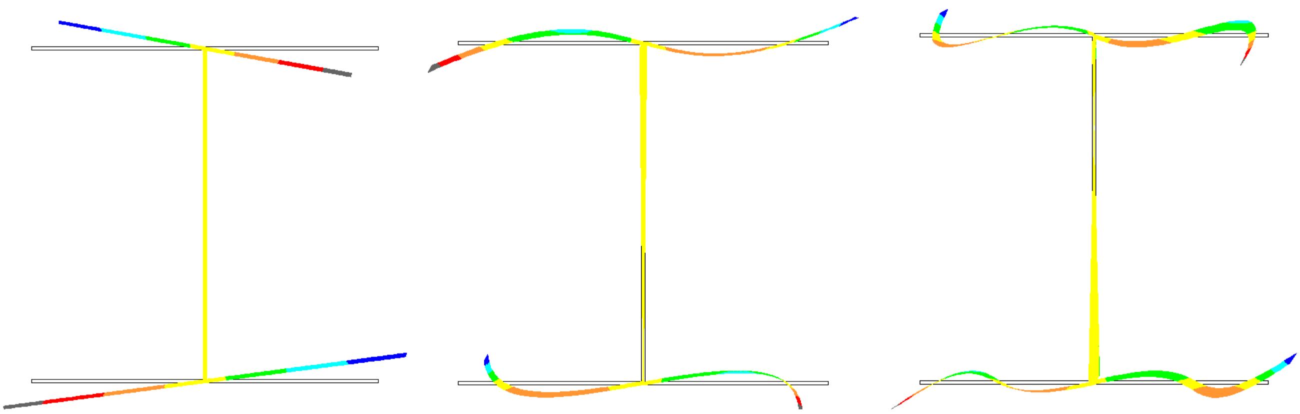
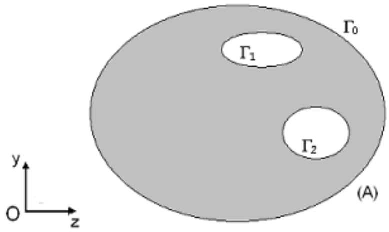


Figure 7 : 3 first warping modes of torsion.

Appendix B:



We will detail in this section a method to solve the partial derivative problem, that we name SF, of the type:

$$\Delta\Omega_{n+1} = \Omega_n \text{ on A} \quad \text{a.1}$$

$$\frac{\partial\Omega_{n+1}}{\partial n} = 0 \quad \text{on } \Gamma \quad \text{a.2}$$

$$\Omega_{n+1} = 0 \quad \text{in a section point} \quad \text{a.3}$$

Let Ω_{n+1} be a solution of SF, and $f \in C^1$ a continious and a derivable real function. We can write then:

$$\int_A (\Delta\Omega_{n+1} - \Omega_n)\, f\, dA = 0 \quad \text{a.4}$$

We use the Green identity to obtain:

$$\int_A (\Delta\Omega_{n+1} - \Omega_n)\, f\, dA = -\int_A \Omega_n\, f\, dA - \int_A \nabla\Omega_{n+1} \cdot \nabla f\, dA + \int_\Gamma f\, \nabla\Omega \cdot n\, d\Gamma = 0 \quad \text{a.5}$$

Where n is the normal vector at a boundary point, ∇ the gradient operator, and $\cdot$ the dot product.
Using the boundary condition a.2, we can write the weak form, WF, of the problem SF:

$$\int_A \nabla\Omega_{n+1} \cdot \nabla f\, dA = -\int_A \Omega_n\, f\, dA \quad \text{a.6}$$

Thus we have demonstrated that if Ω_{n+1} is a solution of SF, then a.6 is verified for every $f \in C^1$. We can easily demonstrate the inverse implication.
To solve the weak form WF of the problem, our cross section will be discretized into triangular element, where we suppose that Ω_{n+1} vary linearly. The warping Ω^p_{n+1} in a point p, will be written in function of the the warping Ω^i_{n+1} at the triangle vertices, by using linear shape functions N^p_i :

$$\Omega^p_{n+1} = \sum_{i=1}^{3} N^p_i\, \Omega^i_{n+1} \quad \text{a.7}$$

We note for the following $a(f,g) = \int_A \nabla f \cdot \nabla g\, dA$ and $(f,g) = \int_A f\, g\, dA$, two symetric and bilinear forms.

We replace a.7 into a.6 to obtain:

$$a(\Omega_{n+1}, f) = -(\Omega_n, f)$$

$$a\left(\sum N_i\, \Omega^i_{n+1}, \sum N_i\, f_i\right) = -\left(\Omega_n, \sum N_i\, f_i\right)$$

$$\sum_i \left(\sum_j a\left(N_i, N_j\right)\Omega^j_{n+1} \right) f_i = -\sum_i \left(\Omega_n, N_i \right) f_i \qquad \text{a.8}$$

The relation a.8 is verified for every f, thus:

$$\sum_j a\left(N_i, N_j\right)\Omega^j_{n+1} = -\left(\Omega_n, N_i\right) \quad for \;\; 1 \le i \le 3 \qquad \text{a.9}$$

This equations can be written in a matrix form:

$$\begin{bmatrix} a(N_1,N_1) & a(N_1,N_2) & a(N_1,N_3) \\ & a(N_2,N_2) & a(N_2,N_3) \\ sym & & a(N_3,N_3) \end{bmatrix} \begin{Bmatrix} \Omega^1_{n+1} \\ \Omega^2_{n+1} \\ \Omega^3_{n+1} \end{Bmatrix} = - \begin{Bmatrix} (N_1,\Omega_n) \\ (N_2,\Omega_n) \\ (N_3,\Omega_n) \end{Bmatrix}$$

To calculate the integrals $a\left(N_i, N_j\right)$ and $\left(N_i, \Omega_n\right)$, we can use a numerical integration method, such as the gaussian quadrature. After assembling the equations a.8 for all the triangular elements of the section mesh, we obtain an equation system, wich solution gives the warping value at each node. This warping map is not yet the one desired, we have to perform the Gram-Schmidt orthogonalization process, to finally obtain the n+1th warping mode.

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